ACTUARIAL STUDIES NO. 4

GRADUATION

OF

MORTALITY AND OTHER TABLES

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GENERAL INTRODUCTION.

In view of the fact that, with the exception of a very few modern text books, the literature of Actuarial Science is contained in scattered original papers, The Actuarial Society of America proposes to issue a series of small volumes upon important actuarial subjects. Each volume is intended to bring together, as far as space permits, the more important points of information on the subject discussed. The objects in issuing the series are twofold: (1) to assist students of Actuarial Science, and (2) to furnish a means of ready reference for Actuaries. The various subjects are allocated to Fellows of the Society by the Committee in Charge; and, associated with the principal contributor, who is primarily responsible for the matter included and the views expressed, are one or more "Associate Contributors." These are appointed for the purpose of aiding and criticizing the work before publication. It is proposed to avoid discussing subjects already covered in the Text Book of the Institute of Actuaries except as continuity of thought may make occasional references necessary. The title chosen to represent the character of this series is "Actuarial Studies."

The thanks of the Society and of the Committee in Charge are due to all the contributors who have freely given of their time and labor, with the sole purpose of helping others—especially students.

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PREFACE.

The following study is intended for the assistance of students preparing for the examinations of the Actuarial Society. The principal contributor desires to acknowledge his indebtedness to the Associate Contributor, and also to Messrs. R. D. Murphy and H. H. Wolfenden for a critical reading of the manuscript, and valuable suggestions with regard to arrangement and wording. It is perhaps appropriate to add here that the references in the text to Sir George F. Hardy's lectures on the "Construction of Tables of Mortality, etc." might have been much more numerous than they appear, more particularly in that part of the study relating to "Graduation by Mathematical Formula."

R. H.

DEC. 6, 1918.

GRADUATION

OF MORTALITY AND OTHER TABLES.

INTRODUCTION.

1. There are two general classes of statistical tables of which, by the nature of things, one only can be the subject of graduation. In tables of one kind, the groups, regarding which information is given, are related to one another merely as parts of some larger group. Examples of statistical tables of this kind are very frequently met. The annual reports of companies of various kinds are each a set of statistical tables analyzing the income and disbursements, assets and liabilities and other particulars of the companies' business into component parts. The abstract tables which are published showing the corresponding items for the various companies furnish another example of this class of statistics. Statistics of this kind are evidently not subject to adjustment of the kind ordinarily implied by the term graduation. All that can be done to increase the weight of the statistics is to combine them with other similar statistics which may be homogeneous with them. For example, a statistical table showing the proportionate distribution of the assets of all the companies of a given class combined would be less liable to fluctuation from year to year than a similar table for the assets of one company only. The various groups appearing in statistics of this kind have no serial relation with one another, and may, therefore, be described as *non-serial statistics*, or statistics of attributes.

2. Tables covering data regarding various municipalities included in a given territory are in their usual form examples of non-serial statistics. If, however, the various municipalities are combined into groups according to population or according to their place in any other quantitative series, the resulting tables for these groups will then belong to the other class of statistics, namely, *serial statistics* or statistics of variables as they will form in this case a connected series in which each group bears a special relation to the groups immediately preceding or following it.

Statistics of various kinds for successive calendar years belong to the class of *serial statistics*. In mortality statistics in particular an analysis of death claims by cause of death would be an example of non-serial statistics, whereas an analysis of the same claims by attained age or by duration of insurance would be an example of serial statistics. Tables of the kind just described showing the relative frequency with which different values of the characteristic used as the basis of analysis appear in a given group of individuals are called frequency distributions.

If the events occur only at isolated values of the variable they may be represented graphically by using the variable as abscissa and erecting ordinates proportionate to the frequencies at those points. Such a diagram would be used, for example, to show how many heads appear in a fixed number of tosses of a coin in a series of such experiments. It is customary to render the outline of the diagram more readily visible by joining the ends of the ordinates by straight lines. If, however, we are tabulating the frequency of a continuously varying quantity, we obtain the number of occurrences corresponding to an interval and not to isolated points. This may be represented graphically by erecting a series of rectangles, each with its respective interval as a base and enclosing an area proportionate to the frequency for that interval. It is usual to assume that if the number of cases observed were indefinitely increased the frequencies of the successive values would form a continuous series; and the continuous curve drawn through the ends of the ordinates, or such that the ordinate for any value of the characteristic is proportionate to the probability of that value, is known as a frequency curve. Tables of the exposed to risk and of the deaths in a mortality experience, are examples of frequency distributions.

In the examples given above, the range of the possible values of the characteristic is limited in both directions, but in certain cases the range of values may for practical purposes be taken as unlimited. This is especially the case in connection with the statistics of any business enterprise or of a state or country for the successive calendar years, with the additional special feature that at any particular time the complete range of the table cannot by nature of things be known.

3. Many cases arise in connection with the tabulation of statistical data where the important element is not the absolute numbers

in the various groups, but rather the proportion which those numbers in one series bears to the corresponding numbers in a collateral series. In the case of a mortality experience, for example, where the exposed to risk and the deaths are tabulated it is ordinarily as a means of determining the ratio, for each age or group of ages, of the deaths to the exposed, or in other words the rate of mortality. In the case of statistics of this kind it is also natural to assume that the values of these ratios would, if the experience were large enough, form a regular series, and this assumption agrees with the results of actual experience, which show that the ratios arising from a large group of observed facts generally approximate more closely to a regular series than where the group is more limited. Tables of this kind may be called tables of ratios to distinguish them from simple frequency distributions.

Reasons for Graduation.

4. We have stated above with reference to both frequency distributions and to tables of ratios that it is a natural assumption that if the number of cases observed were indefinitely increased the successive terms of the resulting table would exhibit a regular progression. This is a particular case of a general principle which is ordinarily expressed by the Latin proverb "natura non agit per saltum." This assumption is not in any way inconsistent with the fact that the results of any limited experience will exhibit a very irregular progression due to the fluctuations arising from the small numbers involved. As we desire in most cases to secure some guide to the probable future experience in similar cases we must make as near an approach as we can to a table showing the results of an unlimited experience. It is necessary, therefore, to substitute a regular series for the irregular one actually arising. This operation is called graduating the series. (1) (5)*

5. Take, for example, a frequency curve under which the probabilities corresponding to the successive intervals are p_1, p_2, p_3 and so on, and suppose the total number of cases observed is n. The expected number of cases in the successive intervals will then be np_1, np_2, np_3 and so on. Consider then any particular interval for which we will suppose that the corresponding probability is p and consequently the expected number of cases np, the theory of

* The numbers appearing in brackets at the ends of this and other paragraphs refer to the publications listed in the Bibliography at the end of the study.

probability shows that the chance of the actual number being exactly np is usually small, the mean square of the difference between the actual number and the expected being $np(1-p)$. The probability of any particular value x of the actual number of cases is

$$\frac{\lfloor n}{\lfloor x \lfloor n-x} p^x (1-p)^{n-x}.$$

and an analysis of the respective probabilities of the various degrees of departure from the expected shows that there is approximately an even chance that the departure will exceed $\frac{2}{3}\sqrt{np(1-p)}$ (see note), that the probability of the actual number exceeding the expected is approximately equal to the probability of its being less, each being approximately one half, and that a departure in one direction from the expected in a particular group is as likely as not to be followed by a departure in the opposite direction in the next succeeding group. From these principles it follows that the actual series of numbers will be an irregular one, but that the proportion which the irregularities bear to the actual numbers involved will decrease as those numbers increase. The same principles may be shown by parallel reasoning to apply to the case of a series of ratios.

6. The table on page 5 illustrates the irregularity due to small numbers. It represents the experience of the British Offices on female lives insured on the Ordinary Life participating plan excluding the first fifty years of assurance. The columns of the exposed to risk and the deaths are examples of frequency distributions, and the marked irregularity of the figures in the column of deaths is worthy of note. This is reflected in the irregularity of the rates of mortality and in the d_x column of the unadjusted mortality table. The values of q_x do not agree exactly with the values of l_x and d_x, but are based directly on the exposed to risk and deaths.

7. The effect of an increase in the number of cases observed on the regularity of a mortality table is shown by a comparison of the values of q_x in the above table with the corresponding values in the same experience excluding only the first five years and with the experience on male lives excluding the first five years. The table on page 6 shows this comparison for ages 60 to 70 inclusive:

ILLUSTRATIVE MORTALITY EXPERIENCE

Age.	Exposed.	Died.	Colog p_x.	Log l_x.	l_x.	d_x.	q_x.
55	2	0	.0000	3.0000	1000	0	.000
56	4	0	.0000	.0000	1000	0	.000
57	11	0	.0000	.0000	1000	0	.000
58	19	0	.0000	.0000	1000	0	.000
59	31	1	.0142	.0000	1000	32	.032
60	48	1	.0091	2.9858	968	20	.021
61	58	3	.0231	.9767	948	49	.052
62	72	2	.0122	.9536	899	25	.028
63	84	0	.0000	.9414	874	0	.000
64	100	4	.0177	.9414	874	35	.040
65	106	1	.0041	2.9237	839	8	.009
66	114	1	.0038	.9196	831	7	.009
67	129	3	.0102	.9158	824	19	.023
68	132	5	.0168	.9056	805	31	.038
69	136	11	.0366	.8888	774	62	.081
70	135	6	.0197	2.8522	712	32	.044
71	143	12	.0381	.8325	680	57	.084
72	140	10	.0322	.7944	623	45	.071
73	144	11	.0345	.7622	578	44	.076
74	149	6	.0179	.7277	534	21	.040
75	154	16	.0476	2.7098	513	54	.104
76	150	24	.0757	.6622	459	73	.160
77	139	8	.0257	.5865	386	22	.058
78	145	16	.0508	.5608	364	40	.110
79	140	13	.0423	.5100	324	30	.093
80	137	19	.0648	2.4677	294	41	.139
81	136	21	.0728	.4029	253	39	.154
82	126	23	.0875	.3301	214	39	.183
83	126	26	.1004	.2426	175	36	.206
84	109	26	.1184	.1422	139	33	.239
85	91	23	.1265	2.0238	106	27	.253
86	77	21	.1383	1.8973	79	22	.273
87	66	16	.1206	.7590	57	14	.242
88	54	12	.1091	.6384	43	9	.222
89	49	15	.1587	.5293	34	11	.306
90	39	9	.1139	1.3706	23	5	.231
91	31	7	.1112	.2567	18	4	.226
92	27	6	.1091	.1455	14	3	.222
93	22	7	.1663	.0364	11	3.6	.318
94	15	2	.0622	0.8701	7.4	1.0	.133
95	12	3	.1249	.8079	6.4	1.6	.250
96	8	4	.3010	.6830	4.8	2.4	.500
97	4	1	.1249	.3820	2.4	.6	.250
98	3	2	.4771	.2571	1.8	1.2	.667
99	1	1	∞	$\bar{1}.7800$	.6	.6	1.000

COMPARATIVE RATES OF MORTALITY

Age.	Values of q_x.		
	$O^{F(50)}$.	$O^{F(5)}$.	$O^{M(5)}$.
60	.021	.026	.030
61	.052	.028	.033
62	.028	.029	.037
63	.000	.032	.037
64	.040	.031	.038
65	.009	.035	.049
66	.009	.038	.047
67	.023	.044	.045
68	.038	.049	.060
69	.081	.054	.061
70	.044	.054	.064

8. It is a general principle of science that in seeking to explain an observed set of facts that hypothesis is adopted which, while being consistent with the general results of collateral observations and with the general principles of nature, best explains the observed facts, or in other words makes their probability a maximum. For example, if we observed that of 1,000 people of a given age, eight die within a year, and if we have no collateral observations at adjacent ages with which it is necessary to harmonize these results, we adopt the hypothesis that the probability of death within one year at that age is 8/1,000 or 1/125, because that is the value of the probability which gives the greatest value to the chance of exactly eight people dying out of 1,000 exposed to risk. Where, however, we have a series of observations at consecutive ages it is necessary to substitute a smooth series for the irregular one representing the ungraduated observations. The substituted series must, from the nature of things, be the result of a compromise between the two factors of smoothness and closeness to observed facts. It is theoretically possible to assign a basis for the numerical measurement of the irregularity of a series as well as for its departure from the observed facts, and by assigning the proportion in which an increase in the one is to be taken as counterbalancing a decrease in the other, to arrive by a mathematical process at the series which best harmonizes the two factors. On any basis suggested, however, the resulting equations are numerous and unwieldy to such an extent as to render the process practically prohibitive. Tentative processes are therefore necessary. These methods come under four general divisions. Under the first method a diagram is

made to represent graphically the observed facts and a continuous curve is then drawn as a basis for the graduated series. Under the second method, the graduated series is formed by interpolation on the basis of values determined for fixed intervals, these values being so determined as to give an interpolated series fitting as closely as possible to the observed facts. Under the third method the individual terms of the graduated series are each determined by a summation of adjacent terms of the original series, a correction being introduced to allow for the differences of the second and higher orders in the series. Under the fourth method a mathematical formula containing arbitrary constants is used to express the series and the constants determined so as to adhere as closely as possible to the observed facts.

9. It is evident that the resulting series even after graduation by any of these methods is still based on a limited number of observations and therefore is necessarily affected by errors arising from such limitation. The result of the graduation, however, is to distribute the errors arising at any particular term over a considerable range of adjacent terms, thus largely reducing the remaining errors by allowing positive and negative errors to counterbalance one another.

Criteria of a Good Graduation.

10. After a graduated series has been constructed, it is usually tested with respect to the two points of smoothness and closeness to the observed facts. With respect to smoothness the fact that a series is determined by a mathematical formula is usually taken as a sufficient test, but when it is not so determined the criterion usually adopted in this respect is the smallness of the third differences in the graduated series. This smallness is sometimes tested by inspection of the differences after they have been taken out, but in comparing two different graduations of the same series, if it is desired to have a numerical measure of the smoothness of each, the sum of the squares of the third differences in the different sections of the table may be taken as such measure.

11. With respect to closeness to the observed facts, the requirements usually made are (1) that the total number and the first and second moments* about any assigned point shall be approxi-

* The nth moment about any assigned point is the sum of the nth powers of the values of the variable measured from that point as origin each multi-

mately the same in the graduated series as in the ungraduated, and (2) that the departures in individual groups shall not on the average materially exceed in magnitude those expected in accordance with the theory of probability. In the case of mortality tables these comparisons are usually based on the expected deaths as obtained by multiplying the number exposed to risk by the rate of mortality at the individual ages. The comparison is usually made by recording the difference between the actual deaths at individual ages and the expected. A continuous summation with due regard to sign of these deviations is then made. The smallness of the numbers in this column of accumulated deviations, the frequency in change of sign and the extent to which positive and negative terms balance one another form the tests of the closeness of agreement in the total number and the first and second moments. This follows from the fact that the successive moments may each be expressed in terms of summations, so that if the summations, up to a certain order, in two series are equal, all moments which may be expressed in terms of those summations are also equal. The number of summations required is one more than the highest order of moments, since the first summation merely determines the total number. (4) (5)

12. From the principles of the theory of probability, it is known that where p is the probability of an event happening and q the probability of its not happening at a given trial, the average deviation from the mean irrespective of sign in n trials is approximately $\frac{4}{5}\sqrt{npq}$ when n is large while the mean value of the square of the departure in the same number of trials is npq. The magnitude of the individual departures can, therefore, be tested by comparing the sum of the individual departures irrespective of sign with its expected value, as derived from the above formula, or the comparison may be based either on the sum of the squares of the departures or on the sum of those squares each divided by its mean

plied by the proportionate frequency of the value. In the language of the infinitesimal calculus the nth moment about the point $x = a$ is

$$\frac{\int_{-\infty}^{+\infty} y(x-a)^n dx}{\int_{-\infty}^{+\infty} y dx}$$

when y represents the relative frequency of the value x. (See also articles 57 to 60 inclusive.)

value. The latter test is the one used by Prof. Karl Pearson to measure the goodness of fit where a frequency curve has been applied to the graduation of statistical tables. Its use is supported logically by the fact that the quantity so arrived at is approximately proportional to the logarithm of the ratio between the probability, on the basis of the graduated table, of the observed facts and that of the results expected according to that table.

Graphic Method.

13. The graphic method of graduating statistical tables arises naturally from the graphic method of representing the tables. It applies to frequency distributions as well as to tables of ratios. Under this method the items of a table are represented by points in a diagram. For convenience in plotting the diagram accurately ruled section paper is ordinarily used, the number of squares measured along a selected base line being taken to represent the argument of the table and the value of the function being represented on a suitable scale by the distance of the point from the base. When the points corresponding to the successive values of the argument are plotted and joined by straight lines it is found that the result is a zigzag line full of minor irregularities, but showing indications, the strength of which depends on the volume of the observations, of an underlying regular law. The graduation of the series is effected by drawing among these points, but not necessarily through any of them, a regular curve representing this law. Preliminary groupings, not necessarily covering equal intervals but so arranged as to produce the greatest attainable regularity, may be made in order to bring out this law. In some cases where the observations are few two different groupings of the same material may indicate different curves. In that case collateral information should be used to determine which to follow. After the curve is drawn the values of the ordinates are read off and the results corrected to remove any irregularities due to errors in reading. A comparison is then made between the graduated series and the original data and the series is amended, if necessary, to bring the variation between the two within the limits considered permissible.

14. In the case of a mortality table this comparison is ordinarily made by computing the expected deaths by the graduated table

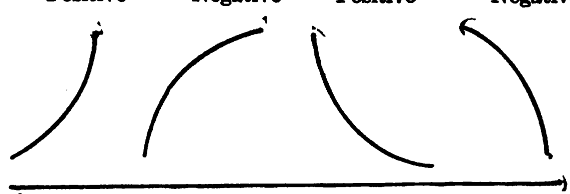
and recording the difference with due regard to sign, between the actual deaths and the expected. The deviations for the successive ages are then summed continuously, forming a column of accumulated deviations. If a relatively large and persistent deviation in either direction is accumulated in any section of the table, the series is amended to reduce or eliminate it.

15. The following extracts from Dr. Sprague's description of the application of this method will indicate the considerations which arise. (10)

". . . In order to understand and successfully apply the graphic method of graduation, it is necessary to study carefully the relations that exist between the progression of the numbers and their differences, and the form of the curve. If we have any smooth curve, and take the values of the ordinates at equal small intervals along the base line, and then find the first and second differences of these values, we may easily satisfy ourselves that, if the first differences are all positive, the curve continually recedes from the base line: and if they are negative, the curve approaches that line; also that if the second differences are positive, the curve has its convexity turned to the base line; and if they are all negative, it is concave to that line.

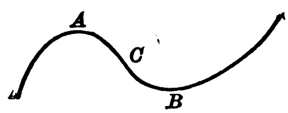
First Differences.	Positive	Positive	Negative	Negative
Second "	Positive	Negative	Positive	Negative

Form of the curve

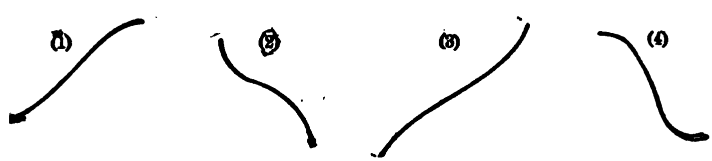


There are thus four cases, which are represented in the above woodcut. As a specimen of the first, we may take the series formed by the squares or cubes of the natural numbers; for the second, we may take the series formed by the square or cube roots of those numbers; for the third, the series formed by the reciprocals, either of the numbers or of their squares or cubes; and those students who are not already familiar with the propositions I have stated, cannot do better than verify them by taking numerical examples. It will be found an instructive exercise to extract from Barlow's very useful tables, numbers of the kind I have described, then to calculate their first and second differences, and lastly to plot down on cross-ruled paper the curve that corresponds to each different series of numbers. As a specimen of the fourth form of curve, we may take the series of numbers obtained by subtracting the squares of the natural numbers from a fixed number; for instance, by subtracting the squares of 1, 2, 3, . . ., 9 from 100, we get the series of numbers 99, 96, 91, 84, 75, 64, 51, 36, 19: the first differences of these are, − 3, − 5, − 7, − 9, − 11, − 13, − 15, − 17; and the second differences are all equal to − 2."

"If we find that the series of first differences changes from positive to negative, then the curve, which was receding from the base line, has a maximum point, and begins to approach that line; and, on the contrary, if the change is from negative to positive, the curve, which was approaching the base line has a minimum point, and begins to recede from that line. These two forms of the curve are shown in the appended woodcut, *A* and *B* being the maximum and minimum points respectively. The student will find several examples of this kind in the two series of graduated probabilities given by Mr. Higham on pages 20 and 21 of his paper."



"If the second differences change sign, the curve has what is called a point of inflection, or a point of contrary flexure; that is to say, up to a certain point it is convex to the base line, and at that point it changes its direction and becomes concave, or the contrary. If the second differences change sign from positive to negative, the curve is first convex to the base line and afterwards concave, as is shown in Figures (1) and (2) of the following woodcut; while if the differences change from negative to positive, the curve passes from concave to convex, as shown in figures (3) and (4) of the woodcut. Many examples of this kind occur in Mr. Higham's second series of adjusted probabilities."



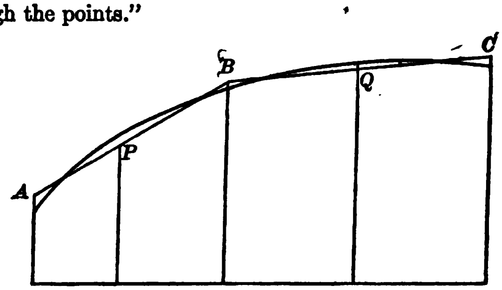
"The student will easily be able to satisfy himself that there must always be a point of inflection between a maximum and a minimum point in the curve. In the neighborhood of a maximum point the curve is concave to the base line, and the second differences are negative; while in the neighborhood of a minimum point the curve is convex to the base line, and the second differences are positive. Hence, in passing from a maximum point to a minimum, the second differences change sign from negative to positive, which proves that there is a point of inflection between them. Thus, in the diagram above, there is a point of inflection, *C*, between the maximum point, *A*, and the minimum point, *B*."

"When, . . . the intervals at which the ordinates are taken are not all equal, all that has been said regarding first differences still holds good, but the statement requires modification as regards the second differences. When the interval is constant, the first difference forms a measure of the rate at which the curve recedes from the base line (or approaches it); but when the intervals are of unequal magnitude, it is clear that, in order to get a proper

measure of the rate at which the curve recedes from (or approaches to) the base line, we must divide each first difference by the interval to which it relates. . . . These numbers, then, which I call divided first differences, measure the rate at which the curve recedes from (or approaches to) the base line. So long as this rate increases, the curve is convex to the base line; and when this rate diminishes, the curve is concave to the base line. We have therefore to take the differences of the divided first differences, . . . and when these are positive, the curve is convex to the base line; and when they are negative, the curve is concave. The number of changes of sign, therefore, indicates the number of points of inflection in the curve. . . ."

"Pursuing the same course with the figures . . . which relate to ages over 70, we see that the final column exhibits 15 changes of sign, corresponding to an equal number of points of inflection in the curve; and . . . we have now to substitute for this irregular curve one which shall present no points of inflection, so that when we take the corresponding probabilities of dying at each age and form the second differences, these shall exhibit no changes of sign. It is here that the graphic method comes in. We have . . . 22 points marked on our sheet of cross-ruled paper; and we must actually draw the best curve we can, that shall pass as near those points as practicable, above some and below others, but without exhibiting any point of inflection."

"I may here remark that, when our points indicate a curve free from sudden changes of direction, and with its curvature in the same direction throughout, theoretical considerations teach us that the curve should not pass exactly through the points, but a little outside them. It does not seem desirable to give here a formal proof of this proposition; but, for the benefit of those who may be disposed to investigate the matter for themselves, it may suffice to state that, having regard to the way in which our points are obtained, if we assume that the rate of mortality increases by constant first differences through each of the intervals we have dealt with, the unadjusted mortality curve will form a polygonal figure ABC . . . the sides of which will be bisected by our points, P, Q, . . .; and our adjusted curve must be drawn so that its area is approximately the same as that of the polygon, as shown in the subjoined figure. The distance, however, between the curve and the points is so small, that it may be generally disregarded, and the curve may be drawn through the points."



". . . The next step is to note the points in which it cuts the vertical lines corresponding to the various ages, and to estimate the length of the

ordinate for each age. . . . There is always a liability to error in estimating the tenths of an interval and small errors may also arise from inequalities in the ruling of the paper, or from the curve being drawn unsteadily. In order to remove these errors I difference the quantities, and, when I find the series of differences presents irregularities, I remove these by inspection."

In applying the graphic method to select or analyzed data Dr. Sprague first formed a short mortality table covering five years from entry for each age at entry. He then averaged the values of $l_{[x]+t}$ and $d_{[x]+t}$ for quinquennial intervals of $[x]$, thus forming values of $q_{[x]+t}$ for quinquennial values of $[x]$. After preliminary elimination of great irregularities from each series he then graduated graphically the values of

$$q_{[x]}, \quad \frac{q_{[x]+1}}{q_{[x]}} \quad \text{and} \quad \frac{q_{[x]+2}}{q_{[x]+1}}.$$

The values of $q_{[x]+3}$ and $q_{[x]+4}$ were then filled in by interpolation between the values of $q_{[x]}$, $q_{[x]+1}$ and $q_{[x]+2}$ so determined and the values of q_{x+5}, etc., in the ultimate table (see J. I. A., Vol. XXI, p. 229).

16. In the construction of the Carlisle Table, Milne used a graphic process to redistribute into the individual ages the population and deaths upon which the table was based. They were originally arranged in groups by quinquennial intervals up to age 20 and thereafter by decennial intervals. A separate diagram was made for each by constructing a series of rectangles such that the base of each represented an age interval and the area the number in the corresponding group. A curve was then drawn through the tops of these rectangles which was made as smooth as possible consistent with the requirement, not observed in the ordinary graphic method, that the area between the curve and the base should be the same for each interval as that of the corresponding rectangle. The area under the curve within the interval corresponding to each individual year of life, then gave the redistributed number of the population or deaths corresponding to that year. From the figures so obtained the central death rates, and from them the rates of mortality were constructed in the usual way. (8)

17. In applying the graphic method to mortality tables and to some others the difficulty is found that if the scale of the diagram is sufficiently large to permit of accurate reading in one part of

the curve it is so large that in another part the curve crosses the lines representing the ordinates at a very acute angle, thus not only increasing the difficulty of reading but multiplying greatly the effect on the resulting values of a slight deviation in the curve. This difficulty is sometimes met by using different scales in the different sections of the table, but where a mathematical formula can be determined which approximately represents the series it may be used as a basis, either by representing graphically, not the term of the series itself, but its ratio to the corresponding value of the mathematical function or by the process explained below.

18. In the case of frequency distributions the mathematical function most generally useful will be that representing the exponential law of error, viz., $\frac{1}{\sqrt{\pi}}e^{-x^2}$. Let then

$$f(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \quad \text{and} \quad F(x) = \frac{\int_{-\infty}^{x} y\, dx}{\int_{-\infty}^{+\infty} y\, dx}$$

where y is the ungraduated value of the functions corresponding to the value x of the argument. In this connection it is to be noted that tables of $f(x) - \frac{1}{2}$ or of $2f(x) - 1$ are to be found in various treatises on probability and that the determination of the values of $F(x)$ will ordinarily reduce to summation as our knowledge of the value of y is usually in the form of a statement of the successive values of $\int_{x}^{x+1} y\, dx$. For each value of x a value of z is then determined by the equation $f(z) = F(x)$ and the successive differences of z are then graduated in the manner already described. From these graduated differences we proceed successively to graduated values of z, $f(z)$ or $F(x)$ and $\Delta F(x)$ which is the graduated series required. Where tables of $f(x)$ are not available the function $\frac{e^x}{1 + e^x}$ may be used instead, in which case we have the relation $z = \log \frac{Y}{N - Y}$, where N is the total number of cases and Y the number of cases for values of the argument less than x. (9) (11)

19. In the case of mortality tables we may put

$$\log l_x = K - Ax - e^z$$

so that we have $z = \log(K - Ax - \log l_x)$ where K and A are so selected as to make the series of values of z in a general way arithmetic. This may be done either by trial or from the equations

$$\frac{\log l_x + Ax - K}{\log l_{x+t} + A(x+t) - K} = \frac{\log l_{x+t} + A(x+t) - K}{\log l_{x+2t} + A(x+2t) - K}$$

$$= \frac{\log l_{x+2t} + A(x+2t) - K}{\log l_{x+3t} + A(x+3t) - K} = \frac{\log l_x - \log l_{x+t} - At}{\log l_{x+t} - \log l_{x+2t} - At}$$

$$= \frac{\log l_{x+t} - \log l_{x+2t} - At}{\log l_{x+2t} - \log l_{x+3t} - At} = \frac{\log l_x - 2\log l_{x+t} + \log l_{x+2t}}{\log l_{x+t} - 2\log l_{x+2t} + \log l_{x+3t}},$$

where x and t are given suitable values. The differences of z are again graduated as before and from these graduated differences are constructed graduated values of z, $\log l_x$, $\log p_x$ and q_x.

COMPARISON OF ACTUAL AND EXPECTED CLAIMS

Age.	Expected Deaths.	Actual Deaths.	Age.	Expected Deaths.	Actual Deaths.
55	.0	0	80	19.0	19
56	.1	0	81	20.4	21
57	.3	0	82	20.5	23
58	.5	0	83	21.4	26
59	.8	1	84	20.7	26
60	1.4	1	85	18.7	23
61	1.8	3	86	17.1	21
62	2.4	2	87	15.8	16
63	3.1	0	88	14.0	12
64	3.9	4	89	13.7	15
65	4.5	1	90	11.7	9
66	5.2	1	91	10.0	7
67	6.3	3	92	9.4	6
68	7.0	5	93	8.2	7
69	7.8	11	94	6.0	2
70	8.4	6	95	5.1	3
71	9.6	12	96	3.7	4
72	10.2	10	97	1.9	1
73	11.4	11	98	1.5	2
74	12.7	6	99	.5	1
75	14.3	16			
76	15.1	24			
77	15.1	8			
78	17.1	16			
79	17.9	13			

20. In graduating the mortality experience given on page 5 the graduated $O^{M(5)}$ table which is constructed by a mathematical formula may be adopted as a basis. The table on page 15 shows the actual and expected deaths, the latter being calculated by multiplying the number exposed by the rate of mortality according to the $O^{M(5)}$ table.

The data may then be grouped in order to reduce the irregularities as follows:

Ages.	Average Age.	Expected.	Actual.	Percentage.
55–63	61.2	10.4	7	67.3
64–69	66.9	34.7	25	72.0
70–74	72.2	52.3	45	86.0
75–79	77.1	79.5	77	97.0
80–84	82.0	102.0	115	112.7
85–92	88.0	110.4	109	98.7
93–99	94.7	26.9	20	74.3

In calculating the average age for each group, each age is weighted in proportion to the expected deaths.

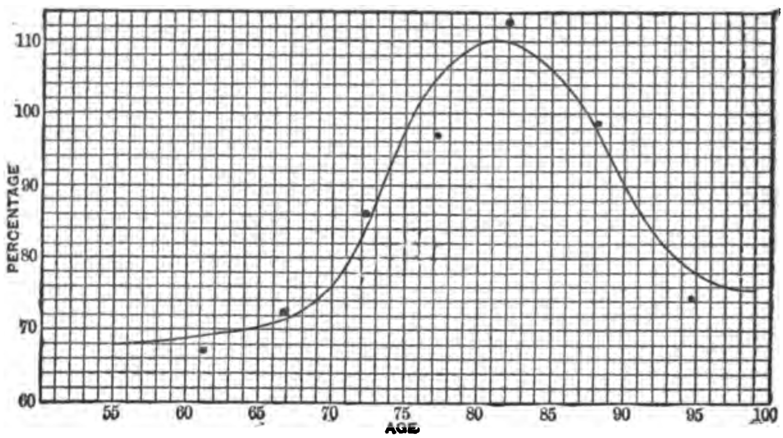
21. We have thus seven points on the diagram to indicate the general course of the curve. They indicate a rapid increase up to a maximum somewhere in the age group 80–84 followed by a rapid decrease. Following these indications a curve is then drawn somewhat freely, especially towards the extremities where the figures involved are small. The percentage corresponding to each age is then read off and the graduated value of q_x computed by taking this percentage of the corresponding value of q_x according to the $O^{M(5)}$ table. The new expected deaths are then computed as shown in the table on page 17.

22. This comparison shows expected deaths to the number of 398.5 compared with 398 actual deaths, the sum of the positive deviations being 46.2 and that of the negative 45.7. The total of the positive accumulated deviations is 61.5 and that of the negative 71.4 and there are eight changes of sign. The sum of the individual deviations irrespective of sign is 91.9, which agrees fairly well with the theroetical amount. The graduation therefore satisfies the tests regarding agreement with the original data. An examination of the values of q_x shows that there is no change in the sign of the first differences, and that the second differences are negative for only ages 86 to 88 and age 96. Slight adjustments were made by inspection above age 90 to eliminate irregularities

GRADUATION OF DATA BY GRAPHIC METHOD

Age.	Percentage.	Graduated q_x	Expected Deaths.	Actual Deaths.	Deviation.	Accumulated Deviation.
55	67.7	.0141	.0	0	.0	.0
56	67.8	.0151	.1	0	+ .1	+ .1
57	67.9	.0161	.2	0	+ .2	+ .3
58	68.1	.0173	.3	0	+ .3	+ .6
59	68.3	.0186	.5	1	− .5	+ .1
60	68.6	.0200	1.0	1	.0	+ .1
61	68.9	.0216	1.2	3	− 1.8	− 1.7
62	69.3	.0234	1.7	2	− .3	− 2.0
63	69.6	.0253	2.2	0	+ 2.2	+ .2
64	69.9	.0273	2.7	4	− 1.3	− 1.1
65	70.3	.0297	3.2	1	+ 2.2	+ 1.1
66	70.8	.0322	3.7	1	+ 2.7	+ 3.8
67	71.5	.0352	4.5	3	+ 1.5	+ 5.3
68	72.3	.0384	5.1	5	+ .1	+ 5.4
69	73.8	.0424	5.8	11	− 5.2	+ .2
70	75.8	.0471	6.4	6	+ .4	+ .6
71	78.8	.0530	7.6	12	− 4.4	− 3.8
72	82.6	.0602	8.4	10	− 1.6	− 5.4
73	87.2	.0689	9.9	11	− 1.1	− 6.5
74	92.6	.0792	11.8	6	+ 5.8	− .7
75	97.5	.0904	13.9	16	− 2.1	− 2.8
76	101.6	.102	15.3	24	− 8.7	−11.5
77	104.5	.114	15.8	8	+ 7.8	− 3.7
78	106.9	.126	18.3	16	+ 2.3	− 1.4
79	108.5	.139	19.4	13	+ 6.4	+ 5.0
80	109.4	.152	20.8	19	+ 1.8	+ 6.8
81	109.8	.165	22.4	21	+ 1.4	+ 8.2
82	109.7	.178	22.5	23	− .5	+ 7.7
83	108.7	.191	23.3	26	− 2.7	+ 5.0
84	107.4	.204	22.2	26	− 3.8	+ 1.2
85	105.6	.217	19.8	23	− 3.2	− 2.0
86	103.3	.230	17.7	21	− 3.3	− 5.3
87	100.8	.242	15.9	16	− .1	− 5.4
88	97.9	.253	13.7	12	+ 1.7	− 3.7
89	94.0	.262	12.9	15	− 2.1	− 5.8
90	90.1	.271	10.5	9	+ 1.5	− 4.3
91	86.8	.280	8.7	7	+ 1.7	− 2.6
92	83.8	.290	7.9	6	+ 1.9	− .7
93	81.3	.302	6.7	7	− .3	− 1.0
94	79.3	.317	4.8	2	+ 2.8	+ 1.8
95	78.0	.335	4.0	3	+ 1.0	+ 2.8
96	77.0	.353	2.8	4	− 1.2	+ 1.6
97	76.2	.368	1.4	1	+ .4	+ 2.0
98	75.8	.384	1.1	2	− .9	+ 1.1
99	75.6	.403	.4	1	− .6	+ .5
			398.5	398	+46.2 −45.7	+61.5 −71.4

arising from the fact that the values of q_x in the table used as a basis were not calculated directly from the formula but from values of l_x and d_x which were recorded to only a small number of figures.



Graduation by Interpolated Series.

23. Milne's method of redistributing the population and deaths described in art. 16 amounts in effect to a graphic interpolation performed upon the function formed by summing continuously the terms to be redistributed. The method of interpolation by means of a mathematical formula has also been applied in various ways to the graduation of mortality tables and other series.

24. In the construction by Farr of the English Life Table No. 3 the data used were the population and deaths for individual years of age up to age 4 inclusive, then for quinquennial groups to age 15 and thereafter for decennial groups. The rates of mortality for ages under 5 were constructed direct from the data and retained unadjusted. In the other groups, except that for ages 10 to 15, the assumption was made that the total deaths divided by the total population would give the force of mortality at the middle of the group. For the exceptional group a special adjustment was made, which was based on the result of an analysis of part of the material. This adjustment was considered necessary because the data indicated the occurrence of a minimum in the middle of the group. (12)

25. The force of mortality so calculated for ages $7\frac{1}{2}$ and $12\frac{1}{2}$

were taken as respectively equal to m_7 and m_{12} and the corresponding rates of mortality were computed by the formula $p_x = \frac{2 - m_x}{2 + m_x}$. In the case of the ten-year groups this method gave the force of mortality for integral ages instead of for the middle of the year of age and a special artifice was adopted to obtain the rates of mortality corresponding to those ages. It was assumed that the force of mortality increased in geometrical progression, so that the following relation held $\mu_{x+t} = r^t\mu_x$ where $r^{10} = \mu_{x+10}/\mu_x$. Then

$$\operatorname{colog}_e p_x = \int_0^1 \mu_{x+t}dt = \frac{r-1}{\log_e r}\mu_x,$$

or transforming into common logarithms

$$\operatorname{Colog} p_x = \frac{k^2(r-1)}{\operatorname{Log} r}\mu_x,$$

where $k = \operatorname{Log}_{10} e$. The values of $\log p_x$, for ages 3, 4, 7, 12, and decennially thereafter, so obtained were used as the basis of third difference interpolation after dividing the table into sections for this purpose. The points of division for the male life table were taken at ages 7, 20 and 50. (12)

26. It has been pointed out that the assumption that the ratio of total deaths to total population in a group represents the force of mortality at the middle of the group considerably understates the mortality at advanced ages and if adopted in connection with a life insurance company's experience somewhat overstates it at the younger ages. Some correction is therefore necessary before the method can be applied with safety and Mr. G. King has devised a method which incorporates the principle of interpolation more fully than Farr's method just described. This method may be applied either to census returns of population and deaths or to the experience of insured lives. (13)

27. The first step of the process is to arrange the population, or the exposed to risk, and the deaths into five-year groups. In the case of an insurance experience there is ordinarily no difficulty in doing so, as the figures are usually available for each year of age. The points of division, however, are not necessarily exact multiples of five years but are so chosen as to facilitate the subsequent interpolations. In the case of census returns the figures are frequently

given for ten-year groups only at mature ages and it is necessary to divide each group into two. This is done by forming tables of T_x, the total population, and of λ_x, the total deaths at age x and over for the values of x which form the points of division between the groups for which the figures are given. Values of these functions for the middle of each group are then calculated by the formula, in which Δ is taken over an interval of ten years.

$$\begin{aligned} U_x &= U_{x-15} + 1.5\Delta U_{x-15} + .375\Delta^2 U_{x-15} - .0625\Delta^3 U_{x-15} \\ &= U_{x-5} + .5\Delta U_{x-5} - .125\Delta^2 U_{x-15} - .0625\Delta^3 U_{x-15} \\ &= \tfrac{1}{2}(U_{x-5} + U_{x+5}) - \tfrac{1}{16}(\Delta U_{x+5} - \Delta U_{x-15}) \\ &= \tfrac{1}{16}\{9(U_{x-5} + U_{x+5}) - (U_{x-15} + U_{x+15})\}. \end{aligned}$$

For the oldest group, where this formula could not be applied, the formula used was in effect $U_x = \frac{1}{4}\{U_{x-15} - 4U_{x-10} + 6U_{x-5} + U_{x+5}\}$ where U_{x-10} is the interpolated value for the middle of the preceding group. In these formulas U_x may be considered as representing Log T_x or Log λ_x as the case may be instead of T_x or λ_x thus reducing the size of the numbers involved. The first differences of the natural numbers corresponding will give the population and deaths in five-year groups.

28. The next step is to calculate the redistributed population or exposed to risk and deaths for the middle year of each group of five. Let w_x represent the total for the group of five ages beginning with age x of either of these numbers and let y_x be a function such that $\Delta y_x = w_x$, where Δ is taken over a five-year interval. Then we have

$$y_{x+n} = y_x + \frac{n}{5}\Delta y_x + \frac{n(n-5)}{50}\Delta^2 y_{x-5} + \frac{n(n^2-25)}{750}\Delta^3 y_{x-5} + \frac{n(n^2-25)(n-10)}{15000}\Delta^4 y_{x-10},$$

whence

$$\begin{aligned} y_{x+2} &= y_x + .4\Delta y_x - .12\Delta^2 y_{x-5} - .056\Delta^3 y_{x-5} + .0224\Delta^4 y_{x-10}, \\ y_{x+3} &= y_x + .6\Delta y_x - .12\Delta^2 y_{x-5} - .064\Delta^3 y_{x-5} + .0224\Delta^4 y_{x-10}, \\ \delta y_{x+2.5} &= .2\Delta y_x - .008\Delta^3 y_{x-5}, \\ &= .2w_x - .008\Delta^2 w_{x-5}, \end{aligned}$$

which is taken as the redistributed value required.

29. From these redistributed values the corresponding rates of mortality, or central death rates as the case may be, are calculated giving values of the function proceeding by quinquennial intervals. In order to complete the table at the oldest ages it is necessary to assume an age at which q_x becomes unity or m_x is equal to 2. This assumption should be made to harmonize with the run of the experience as that limit is approached; the series of quinquennial values is completed by a third difference interpolation based on this value and the three highest reliable values. It is well to fix upon the limiting age in advance if possible and to so arrange the grouping that it will form a term in the series of quinquennial values. For example if 103 is the age selected the groups will run 16–20, 21–25, etc., while if 102 is the age the groups will be 15–19, 20–24, etc. (13) (14) (15)

30. The rates for intermediate ages are then determined by interpolation on these values of q_x or of m_x or on some function of the values, such as $\log (q_x + .1)$, which is more nearly arithmetic, by an osculatory interpolation formula which passes continuously through the successive groups. King suggests the use of Karup's formula which may be written

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{2}\Delta^2 u_{x-1} + \frac{n(n-1)^2}{2}\Delta^3 u_{x-1},$$

where n is less than unity. But this sometimes gives an undulating curve and in that case it will be better to use the more accurate formula,

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{2}\left(\Delta^2 u_x - \tfrac{1}{6}\Delta^4 u_{x-1}\right) + \frac{n(n-1)(n-2)}{6}\left(\Delta^3 u_{x-1} - \tfrac{1}{6}\Delta^5 u_{x-2}\right).$$

In applying this formula at either end of the table where terms are not available for the calculation of the differences required it is assumed that the fifth differences that cannot be computed vanish and the other differences are filled in consistently with that assumption. A practical method of applying this formula is given in the Transactions of the Actuarial Society (Vol. IX., page 211).

31. In applying Mr. King's method to the experience summarized in Art. 6 we have the following totals by five-year age groups:

Ages.	Exposed.	Died.
55–59	67	1
60–64	362	10
65–69	617	21
70–74	711	45
75–79	728	77
80–84	634	115
85–89	337	87
90–94	134	31
95–99	28	11

32. It is evident that the number of deaths in each group is not yet sufficiently large to give a regular progression of rates of mortality. We accordingly seek a formula of greater weight than that used by Mr. King. With the notation used above we have:

$$y_{x+10} = y_x + 2\Delta y_x + \Delta^2 y_{x-5} + \Delta^3 y_{x-5},$$

$$y_{x-5} = y_x - \Delta y_x + \Delta^2 y_{x-5}.$$

Whence

$$y_{x+10} - y_{x-5} = 3\Delta y_x + \Delta^3 y_{x-5},$$

also

$$y_{x+15} = y_x + 3\Delta y_x + 3\Delta^2 y_{x-5} + 4\Delta^3 y_{x-5} + \Delta^4 y_{x-10},$$

$$y_{x-10} = y_x - 2\Delta y_x + 3\Delta^2 y_{x-5} - \Delta^3 y_{x-5} + \Delta^4 y_{x-10}.$$

Whence

$$y_{x+15} - y_{x-10} = 5\Delta y_x + 5\Delta^3 y_{x-5};$$

therefore

$$65(y_{x+10} - y_{x-5}) - 14(y_{x+15} - y_{x-10}) = 125\Delta y_x - 5\Delta^3 y_{x-5} = 625\delta y_{x+2.5}.$$

But

$$y_{x+10} - y_{x-5} = w_{x-5} + w_x + w_{x+5}$$

and

$$y_{x+15} - y_{x-10} = w_{x-10} + w_{x-5} + w_x + w_{x+5} + w_{x+10}.$$

So that

$$625\delta y_{x+2.5} = 51(w_{x-5} + w_x + w_{x+5}) - 14(w_{x-10} + w_{x+10}).$$

33. Applying this formula to the above figures we obtain the results shown below, where E_x and θ_x, respectively, represent Exposed to Risk and Deaths between ages x and $x + 1$.

The value of q_{97} so obtained, however, is evidently unreliable and is rejected, a new value being calculated by interpolation, on the assumption that q_{102} is unity and fourth differences vanish. The formula is

$$q_{102} - 4q_{97} + 6q_{92} - 4q_{87} + q_{82} = \Delta^4 q_{82} = 0$$

or

$$4q_{97} = q_{102} + 6q_{92} - 4q_{87} + q_{82}.$$

The value so obtained is .5111. The value of q_{57} is also unreliable, but is not seriously inconsistent with the other values and is accordingly retained for the purpose of completing the table so as to permit of a complete comparison of actual with expected deaths.

Age.	$625E_x$.	$625\theta_x$.	q_x.
57	12,941	267	.0206
62	43,392	1,002	.0231
67	75,060	2,784	.0371
72	88,912	5,543	.0623
77	92,371	10,575	.1145
82	74,819	13,165	.1765
87	45,771	10,651	.2327
92	16,573	4,969	.2998
97	3,549	884	.2491

34. We have therefore as a basis of interpolation the following values and differences:

x	10^4q_x.	$10^4\Delta q_x$.	$10^4\Delta^2q_x$.	$10^4\Delta^3q_{x-5}$.
57	206	25	115	
62	231	140	112	− 3
67	371	252	270	158
72	623	522	98	−172
77	1145	620	−58	−156
82	1765	562	109	167
87	2327	671	1442	1333
92	2998	2113	2776	1334
97	5111	4889		
102	10000			

Interpolating then by Karup's formula and continuing down to age 55 on the assumption of constant second differences, we obtain the rates of mortality shown in the table on page 24.

Summation Formulas.

35. Before entering upon a discussion of graduation by summation formulas it will be well to explain a system of notation covering the various operations involved and to show in what way the principle of the separation of symbols will apply to the symbols adopted to represent these operations. The fundamental operation involved, in addition to the elementary ones of addition, sub-

Graduation of Data by Interpolation

x.	q_x.	Expected Deaths.	Actual Deaths.	Deviation.	Accumulated Deviation.
55	.0228	.0	0	.0	.0
56	.0215	.1	0	+ .1	+ .1
57	.0206	.2	0	+ .2	+ .3
58	.0202	.4	0	+ .4	+ .7
59	.0202	.6	1	− .4	+ .3
60	.0207	1.0	1	.0	+ .3
61	.0217	1.3	3	− 1.7	− 1.4
62	.0231	1.5	2	− .5	− 1.9
63	.0250	2.1	0	+ 2.1	+ .2
64	.0273	2.7	4	− 1.3	− 1.1
65	.0301	3.2	1	+ 2.2	+ 1.1
66	.0334	3.8	1	+ 2.8	+ 3.9
67	.0371	4.8	3	+ 1.8	+ 5.7
68	.0410	5.4	5	+ .4	+ 6.1
69	.0451	6.1	11	− 4.9	+ 1.2
70	.0497	6.7	6	+ .7	+ 1.9
71	.0554	7.9	12	− 4.1	− 2.2
72	.0623	8.7	10	− 1.3	− 3.5
73	.0709	10.2	11	− .8	− 4.3
74	.0808	12.0	6	+ 6.0	+ 1.7
75	.0916	14.1	16	− 1.9	− .2
76	.103	15.5	24	− 8.5	− 8.7
77	.115	16.0	8	+ 8.0	− .7
78	.126	18.3	16	+ 2.3	+ 1.6
79	.139	19.5	13	+ 6.5	+ 8.1
80	.152	20.8	19	+ 1.8	+ 9.9
81	.164	22.3	21	+ 1.3	+11.2
82	.177	22.3	23	− .7	+10.5
83	.188	23.7	26	− 2.3	+ 8.2
84	.199	21.7	26	− 4.3	+ 3.9
85	.210	19.1	23	− 3.9	− .0
86	.221	17.0	21	− 4.0	− 4.0
87	.233	10.6	16	− 5.4	− 9.4
88	.243	13.1	12	+ 1.1	− 8.3
89	.252	12.3	15	− 2.7	−11.0
90	.262	10.2	9	+ 1.2	− 9.8
91	.277	8.6	7	+ 1.6	− 8.2
92	.300	8.1	6	+ 2.1	− 6.1
93	.328	7.2	7	+ .2	− 5.9
94	.361	5.4	2	+ 3.4	− 2.5
95	.401	4.8	3	+ 1.8	− .7
96	.449	3.6	4	− .4	− 1.1
97	.511	2.0	1	+ 1.0	− .1
98	.587	1.8	2	− .2	− .3
99	.675	.7	1	− .3	− .6
		397.4	398	+49.0 −49.6	+76.9 −92.0

traction, multiplication and division, is that of proceeding to the next term of the series. Let us designate this operation by writing E before the function operated upon. Then we have $EU_x = U_{x+1}$; $E^2U_x = EEU_x = EU_{x+1} = U_{x+2}$ and generally $E^nU_x = U_{x+n}$. It is then evident that $E^mE^nU_x = U_{x+m+n} = E^{m+n}U_x$ so that the exponential law applies in the same way to E as if it were an ordinary quantity. We have also $E(U_x + V_x) = U_{x+1} + V_{x+1} = EU_x + EV_x$ so that the distributive law applies. Also $EaU_x = aU_{x+1} = aEU_x$, where a is a quantity, so that the commutative law applies to the operation in combination with ordinary quantities. Also we may extend the definition and say that generally

$$\begin{aligned}(E + a)U_x = EU_x + aU_x = U_{x+1} + aU_x &= aU_x + U_{x+1} \\ &= (a + E)U_x.\end{aligned}$$

With this understanding it is evident that the operator E may be separated from the function upon which it operates and treated in algebraic transformations as if it were an ordinary quantity. It is to be carefully observed that the function operated upon must not enter into the transformation. In other words the quantities entering into the transformations must be constants and not functions subject to the operator. From the above reasoning it is evident that any other operator which may be expressed in terms of E and constants will possess the same property. Coming under this general class we have the following operations:

$$\begin{aligned}
&\Delta = E - 1, &&\text{or } \Delta U_x = U_{x+1} - U_x, \\
&D = Lt_{h=0}\frac{E^h - 1}{h} = \log_e E, &&\text{or } DU_x = Lt_{h=0}\frac{U_{x+h} - U_x}{h} = \frac{dU_x}{dx}, \\
&\delta = E^{1/2} - E^{-1/2}, &&\text{or } \delta U_x = U_{x+1/2} - U_{x-1/2}, \\
&[\overline{n}] = (E^{n/2} - E^{-n/2}), &&\text{or } [\overline{n}]U_x = U_{x+(n/2)} - U_{x-(n/2)}, \\
&\gamma_n = (E^n + E^{-n}), &&\text{or } \gamma_n U_x = U_{x+n} + U_{x-n}, \\
&[n] = \frac{E^{n/2} - E^{-n/2}}{E^{1/2} - E^{-1/2}} = \frac{[\overline{n}]}{\delta}, &&\text{or } [n]U_x = U_{x+(n-1)/2} + U_{x+(n-3)/2} \\
& && \qquad + \cdots U_{x-(n-1)/2}.
\end{aligned}$$

From the above expansion it will be seen that $[n]U_x$ is the sum of n terms with respect to which U_x is centrally situated. It will be noted that where n is an even number U_x itself does not appear in the summation.

36. For the purpose of demonstrating the possibility of separating the symbols of operation it was convenient to start with E as the fundamental operation and to express the others in terms of it, but for our future purposes it will be convenient to express the various operators in terms of ascending powers of D.

We have

$$D = \log_e E$$

or

$$E = e^D = 1 + D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24} + \cdots,$$

which is readily seen to follow directly from Taylor's series, since

$$\begin{aligned}
EU_x &= U_{x+1} = U_x + \frac{dU_x}{dx} + \frac{1}{2}\frac{d^2U_x}{dx^2} + \frac{1}{6}\frac{d^3U_x}{dx^3} + \frac{1}{24}\frac{d^4U_x}{dx^4} + \cdots \\
&= (1 + D + \tfrac{1}{2}D^2 + \tfrac{1}{6}D^3 + \tfrac{1}{24}D^4 + \cdots)U_x = e^D U_x, \\
\delta &= e^{D/2} - e^{-D/2} = D + \frac{D^3}{24} + \frac{D^5}{1920} + \cdots, \\
E^n &= e^{nD} = 1 + nD + \frac{n^2D^2}{2} + \frac{n^3D^3}{6} + \frac{n^2D^4}{24} + \frac{n^5D^5}{120} + \cdots, \\
[\overline{n}] &= E^{n/2} - E^{-n/2} = nD + \frac{n^3D^3}{24} + \frac{n^5D^5}{1920} + \cdots, \\
\gamma_n &= E^n + E^{-n} = 2 + n^2D^2 + \frac{n^4D^4}{12} + \cdots, \\
[n] &= \frac{nD + n^3D^3/24 + n^5D^5/1920 + \cdots}{D + D^3/24 + D^5/1920 + \cdots} \\
&= n + \frac{n(n^2-1)}{24}D^2 + \frac{n\{5(n^2-1)^2 - 2(n^4-1)\}}{5760}D^4 + \cdots.
\end{aligned}$$

37. An examination of King's method already described will show that in determining the rates of mortality to be used as a basis for the interpolated series, he uses adjusted values of the exposed and of the deaths. These adjusted values are obtained by grouping and redistributing according to the principles of finite differences but the final formula for the adjusted value of U_x expressed in the above notation is

$$\begin{aligned}
U_x' &\doteq \tfrac{1}{5}\{[5]U_x - \tfrac{1}{25}[\overline{5}]^2[5]U_x\} = \frac{[5]}{5}\{1 - \tfrac{1}{25}[\overline{5}]^2\}U_x \\
&= \frac{[5]}{5}(\tfrac{27}{25} - \tfrac{1}{25}\gamma_5)U_x
\end{aligned}$$

since $[\overline{n}]^2$ is seen from its definition to be equal to $\gamma_n - 2$. The process followed in this preliminary adjustment partakes therefore of the nature of a summation formula. Expanding the operators in this last expression in powers of D to the fifth inclusive we have

$$(1 + D^2 + \tfrac{17}{60}D^4 + \cdots)(1 - D^2 - \tfrac{25}{12}D^4 - \cdots) = (1 - \tfrac{14}{5}D^4 - \cdots).$$

38. The first summation method employed was that devised by Mr. Woolhouse. It was originally arrived at by a process of interpolation. Each of the five sets of quinquennial values was made the basis of a second difference interpolation and the intermediate values computed. There were thus obtained for each point five values, one of which was the original value and the remaining four interpolated values. The average of these five values was taken as the graduated value of the function. The general formula for such an interpolation may be written

$$U_x = \frac{(x-b)(x-c)}{(a-b)(a-c)}U_a + \frac{(x-a)(x-c)}{(b-a)(b-c)}U_b + \frac{(x-a)(x-b)}{(c-a)(c-b)}U_c.$$

The expressions for these five values are then as follows:

$$
\begin{array}{ll}
(1) & -\tfrac{3}{25}U_{x-7} + \tfrac{21}{25}U_{x-2} + \tfrac{7}{25}U_{x+3}, \\
(2) & -\tfrac{2}{25}U_{x-6} + \tfrac{24}{25}U_{x-1} + \tfrac{3}{25}U_{x+4}, \\
(3) & U_x, \\
(4) & \tfrac{3}{25}U_{x-4} + \tfrac{24}{25}U_{x+1} - \tfrac{2}{25}U_{x+6}, \\
(5) & \tfrac{7}{25}U_{x-3} + \tfrac{21}{25}U_{x+2} - \tfrac{3}{25}U_{x+7}.
\end{array}
$$

The average of the five values then takes the following form, omitting x from the subscript:

$$
\begin{aligned}
U_0' &= \tfrac{1}{125}\{25U_0 + 24(U_1 + U_{-1}) + 21(U_2 + U_{-2}) + 7(U_3 + U_{-3}) \\
&\qquad + 3(U_4 + U_{-4}) - (2(U_6 + U_{-6}) - 3(U_7 + U_{-7})\} \\
&= \tfrac{1}{125}\{25 + 24\gamma_1 + 21\gamma_2 + 7\gamma_3 + 3\gamma_4 - 2\gamma_6 - 3\gamma_7\}U_0. \quad (16)
\end{aligned}
$$

39. It was later discovered, however, that a graduation by Woolhouse's formula could be effected by means of a summation process. If we designate by G_w the operation of graduating by this formula, we have

$$
\begin{aligned}
125G_wU_0 &= 25U_0 + 24(U_1 + U_{-1}) + 21(U_2 + U_{-2}) \\
&\quad + 7(U_3 + U_{-3}) + 3(U_4 + U_{-4}) - 2(U_6 + U_{-6}) - 3(U_7 + U_{-7}).
\end{aligned}
$$

But it can readily be seen by actual expansion that

$$\begin{aligned}[5]^3U_0 &= (E^2 + E + 1 + E^{-1} + E^{-2})^3U_0 \\ &= 19U_0 + 18(U_1 + U_{-1}) + 15(U_2 + U_{-2}) + 10(U_3 + U_{-3}) \\ &\quad + 6(U_4 + U_{-4}) + 3(U_5 + U_{-5}) + (U_6 + U_{-6}).\end{aligned}$$

So that

$$\begin{aligned}125G_wU_0 &= [5]^3U_0 + 6(U_{-2} + U_{-1} + U_0 + U_1 + U_2) \\ &\quad - 3(U_{-7} + U_{-6} + U_{-5} + U_{-4} + U_{-3}) \\ &\quad - 3(U_3 + U_4 + U_5 + U_6 + U_7) \\ &= [5]^3U_0 - 3[5]^2[5]U_0 = [5]^3U_0 - 3\delta^2[5]^3U_0 \\ &= [5]^3(1 - 3\delta^2)U_0 = [5]^3\{10 - 3[3]\}U_0,\end{aligned}$$

since $[3] = 3 + \delta^2$. Thus

$$G_w = \tfrac{1}{125}[5]^3(1 - 3\delta^2) = \tfrac{1}{125}[5]^3\{10 - 3[3]\}.$$

It will be noticed that the formula calls for a final division by 125, but the number of figures involved in the summations can be reduced by dividing by 10 after the first summation in fives so that the working formula becomes $\frac{2}{25}[5]^2\frac{1}{10}[5]\{10 - 3[3]\}$. (17) (18)

40. The schedule on page 29 shows the actual work of applying this formula to the graduation of a series of values of q_x.

41. Expanding G_w in ascending powers of D to the fourth inclusive we have

$$\begin{aligned}G_w &= \tfrac{1}{125}[5]^3(1 - 3\delta^2) = \tfrac{1}{125}[5]^3\{10 - 3[3]\} \\ &= \tfrac{1}{125}(5 + 5D^2 + \tfrac{17}{12}D^4 + \cdots)^3(1 - 3D^2 - \tfrac{1}{4}D^4 - \cdots) \\ &= (1 + D^2 + \tfrac{17}{60}D^4 + \cdots)^3(1 - 3D^2 - \tfrac{1}{4}D^4 - \cdots) \\ &= (1 + 3D^2 + \tfrac{77}{20}D^4 + \cdots)(1 - 3D^2 - \tfrac{1}{4}D^4 - \cdots) \\ &= (1 - 5{\cdot}4D^4 \cdots).\end{aligned}$$

This shows that a series such that the fourth and higher differential coefficients vanish will be exactly reproduced and that a series for which those coefficients are relatively small will be approximately reproduced. The assumption underlying a graduation by such a formula as this is therefore that within the range covered by the formula the curve representing the true values of the function may be represented by a parabola of the third order.

TABLE ILLUSTRATING WOOLHOUSE'S FORMULA

x.	(1) $10^5 q_x$.	(2) [3] (1).	(3) 3 (2).	(4) 10 (1)–(3).	(5) $\frac{1}{10}$ [5] (4).	(6) [5] (5).	(7) [5] (6).	(8) $\frac{2}{25}$ (7).	
23	511								
24	633	1,830	5,490	840					
25	686	1,963	5,889	971					
26	644	1,932	5,796	644	332				
27	602	1,711	5,133	887	266				
28	465	1,556	4,668	−18	300	1,267			
29	489	1,573	4,719	171	225	1,305			
30	619	1,625	4,875	1,315	144	1,420	7,065	565	563
31	517	1,757	5,271	−101	370	1,444	7,704	616	612
32	621	2,047	6,141	69	381	1,629	8,281	662	669
33	909	2,283	6,849	2,241	324	1,906	8,724	698	707
34	753	2,414	7,242	288	410	1,882	9,329	746	745
35	752	2,258	6,774	746	421	1,863	9,825	786	763
36	753	2,258	6,774	756	346	2,049	10,086	807	828
37	753	2,451	7,353	177	362	2,125	10,528	842	867
38	945	2,653	7,959	1,491	510	2,167	11,229	898	868
39	955	3,035	9,105	445	486	2,324	11,745	940	
40	1,135	3,038	9,114	2,236	463	2,564	12,306	984	
41	948	2,991	8,973	507	503	2,565			
42	908	3,042	9,126	− 46	602	2,686			
43	1,186	3,324	9,972	1,888	511				
44	1,230	3,623	10,869	1,431	607				
45	1,207	3,579	10,737	1,333					
46	1,142	3,320	9,960	1,460					
47	971								

42. Mr. Higham who devised this method of applying the formula suggested a modification which effects a marked improvement in the results. This consisted in substituting $[3] - \gamma_2$ or $2[3] - [5]$ for $1 - 3\delta^2$. Designating his graduation by G_h we have

$$G_h = \tfrac{1}{125}[5]^3\{[3] - \gamma_2\}$$
$$= (1 + 3D^2 + \tfrac{77}{20}D^4 + \cdots)(1 - 3D^2 - \tfrac{5}{4}D^4 - \cdots)$$
$$= (1 - 6.4D^4 - \cdots).$$

The table on page 30 shows the practical application of this formula to the series to which Woolhouse's was applied. (17)

43. Each term U_x of the series to which a summation formula is applied may be considered as made up of two parts V_x the true value of the function which would be arrived at on a sufficiently broad experience and E_x the error or departure from that value so that we have $U_x = V_x + E_x$. Using G then as the general symbol for a graduation to which the distributive law applies we

TABLE ILLUSTRATING HIGHAM'S FORMULA

x.	(1) 10^5q_x.	(2) $[8](1)$.	(3) $\gamma_2(1)$.	(4) (2)–(3).	(5) $\frac{1}{10}[5](4)$.	(6) $[5](5)$.	(7) $[5](6)$.	(8) $\frac{2}{25}(7)$.
22	559							
23	511	1,703						
24	633	1,830	1,203	627				
25	686	1,963	1,113	850				
26	644	1,932	1,098	834	314			
27	602	1,711	1,175	536	297			
28	465	1,556	1,263	293	266	1,327		
29	489	1,573	1,119	454	218	1,317		
30	619	1,625	1,086	539	232	1,383	7,168	573
31	517	1,757	1,398	359	304	1,485	7,660	613
32	621	2,047	1,372	675	363	1,656	8,228	658
33	909	2,283	1,269	1,014	368	1,819	8,767	701
34	753	2,414	1,374	1,040	389	1,885	9,295	744
35	752	2,258	1,662	596	395	1,922	9,751	780
36	753	2,258	1,698	560	370	2,013	10,130	810
37	753	2,451	1,707	744	400	2,112	10,595	848
38	945	2,653	1,888	765	459	2,198	11,168	893
39	955	3,035	1,701	1,334	488	2,350	11,760	941
40	1,135	3,038	1,853	1,185	481	2,495	12,317	985
41	948	2,991	2,141	850	522	2,605		
42	908	3,042	2,365	677	545	2,669		
43	1,186	3,324	2,155	1,169	569			
44	1,230	3,623	2,050	1,573	552			
45	1,207	3,579	2,157	1,422				
46	1,142	3,320	2,642	678				
47	971	3,525						
48	1,412							

have $GU_x = G(V_x + E_x) = GV_x + GE_x$. The effect of the graduation therefore consists of two parts, one being its effect on the smooth series of true values and the other its effect on the errors. As regards the first it is required that the series shall be reproduced as nearly as possible and as regards the second it is required that the errors shall be reduced as far as possible and that the residual errors shall form a continuous series. The assumption is usually made with respect to V_x that fourth and higher differences may be neglected. Summation formulas are therefore generally arranged so that the coefficient of D^2 either vanishes or is very small, a value of 1/12 of the opposite sign to the coefficient of D^4 being sometimes permitted.

44. A summation process of graduation usually consists of three summations, not necessarily over equal intervals, following a preliminary adjustment and followed by a division by the number

necessary to reduce the function to the original scale. The general expression for this operation may therefore be written.

$$G = \frac{[p][q][r]}{pqr}\{1 + 2(a + b + c) - a\gamma_1 - b\gamma_2 - c\gamma_3\}.$$

But we have generally

$$\frac{[n]}{n} = 1 + \frac{n^2 - 1}{24}D^2 + \frac{5(n^2 - 1)^2 - 2(n^4 - 1)}{5760}D^4 + \cdots,$$

so that to the third power of D inclusive we have

$$G = \left(1 + \frac{p^2 - 1}{24}D^2\right)\left(1 + \frac{q^2 - 1}{24}D^2\right)\left(1 + \frac{r^2 - 1}{24}D^2\right)$$
$$\{1 - (a + 4b + 9c)D^2\}$$
$$= 1 + \left\{\frac{p^2 + q^2 + r^2 - 3}{24} - (a + 4b + 9c)\right\}D^2.$$

In order therefore that the formula may be correct to third differences we must have

$$a + 4b + 9c = \frac{p^2 + q^2 + r^2 - 3}{24}.$$

In order that the graduated values obtained shall correspond to integral values of the variable it is also necessary that $p + q + r$ should be an odd number, so that p, q and r must be all odd numbers or two even and one odd. In Woolhouse's formula p, q and r are each equal to 5, and a is equal to 3, b and c being each zero, so that the relation holds. In Higham's formula a is equal to -1 and b is equal to 1, so that $a + 4b = 3$ and the relation holds. In Hardy's modification of Higham's formula $([4][5][6])/120$ is substituted for $[5]^3/125$ so that we have

$$G = 1 + \left\{\frac{16 + 25 + 36 - 3}{24} - 3\right\}D^2 \cdots = 1 + \tfrac{1}{12}D^2 \cdots,$$

showing a second difference error in this case. In Karup's 19-term formula the summations are in fives and the values of a, b and c are $-\frac{3}{5}$, 0 and $\frac{2}{5}$ respectively, so that $a + 4b + 9c = \frac{18}{5} - \frac{3}{5} = 3$, and the relation holds. The expression for this formula is then

$$G = \frac{[5]^3}{125}(\tfrac{3}{5} + \tfrac{3}{5}\gamma_1 - \tfrac{2}{5}\gamma_3) = \frac{[5]^3}{625}\{3[3] - 2\gamma_3\}.$$

In Spencer's 21-term formula two summations in fives and one

in sevens are used and $(p^2 + q^2 + r^2 - 3)/24$ becomes

$$\frac{25 + 25 + 49 - 3}{24} = 4,$$

also $a = -\frac{1}{2}$, $b = 0$ and $c = \frac{1}{2}$ so that $a + 4b + 9c = \frac{9}{2} - \frac{1}{2} = 4$, and the formula is correct to third differences. This formula may be expressed symbolically as follows:

$$G = \frac{[5]^2[7]}{175}(1 + \tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma_3) = \frac{[5]^2[7]}{350}(2 + \gamma_1 - \gamma_3)$$

$$= \frac{[5]^2[7]}{350}\{1 + [3] - \gamma_3\}.$$

The following table shows the method given by Spencer for applying the formula to the graduation of a table of q_x: (19)

TABLE ILLUSTRATING SPENCER'S FORMULA

x.	(1) $10^5 q_x$.	(2) $\frac{1}{7}$(1).	(3) [3] (2).	(4) γ_3(2).	(5) (2)+(3)−(4).	(6) [7] (5).	(7) $\frac{1}{5}$(6).	(8) [5] (7).	(9) $\frac{1}{10}$[5] (8).
20	569	81							
21	235	34	195						
22	559	80	187						
23	511	73	243	173	143				
24	633	90	261	120	231				
25	686	98	280	146	232				
26	644	92	276	143	225	1,212	242		
27	602	86	244	178	152	1,173	235		
28	465	66	222	172	116	1,093	219	1,129	
29	489	70	224	181	113	1,066	213	1,131	
30	619	88	232	216	104	1,102	220	1,158	592
31	517	74	251	174	151	1,221	244	1,212	617
32	621	89	293	177	205	1,311	262	1,289	652
33	909	130	327	196	261	1,363	273	1,383	693
34	753	108	345	182	271	1,448	290	1,478	735
35	752	107	323	224	206	1,569	314	1,566	777
36	753	108	323	266	165	1,695	339	1,639	817
37	753	108	351	270	189	1,752	350	1,705	856
38	945	135	379	242	272	1,732	346	1,778	896
39	955	136	433	238	331	1,782	356	1,872	938
40	1,135	162	433	277	318	1,936	387	1,971	979
41	948	135	427	311	251	2,166	433	2,054	
42	908	130	434	308	256	2,245	449	2,114	
43	1,186	169	475	325	319	2,146	429		
44	1,230	176	517	274	419	2,081	416		
45	1,207	172	511	332	351				
46	1,142	163	474	405	232				
47	971	139	504	390	253				
48	1,412	202	577						
49	1,654	236	652						
50	1,498	214							

45. We have hitherto confined our attention largely to the effect of the graduation on V_x and we have seen that a great variety of formulas may be devised which will reproduce, or practically reproduce a curve of the third degree. In order to obtain a complete view of these formulas we must now consider their effect on E_x. In this investigation we will assume that the function to be graduated is such that the successive values of E_x are independent of one another. We will also assume that the mean value of the square of E_x is the same for each term. This is not strictly true but the effect of the assumption will be relatively much the same in the different formulas. It has been already stated that a graduation is expected to reduce as far as possible the values of E_x and to convert what remains of them into as smooth a series as possible. We will therefore investigate the mean value of the square of the graduated value of E_x and the mean value of the square of the third difference of that value.

46. In order to arrive at the effect on the magnitude of E_x it is necessary to expand the graduation formula in terms of E. Any formula may be expressed as follows:

$$\begin{aligned} G &= a_0 + a_1(E + E^{-1}) + a_2(E^2 + E^{-2}) + \cdots \\ &= a_0 + a_1\gamma_1 + a_2\gamma_2 + \cdots. \end{aligned}$$

So that we have

$$GE_x = a_0E_x + a_1(E_{x+1} + E_{x-1}) + a_2(E_{x+2} + E_{x-2}) + \cdots.$$

And, since the values of E_x are independent of one another and the mean value of each is zero, the mean value of the square of GE_x is, if we write μ_2 for the mean value of the square of E_x, $\mu_2(a_0^2 + 2a_1^2 + 2a_2^2 + \cdots)$. The reciprocal of the value of $(a_0^2 + 2a_1^2 + 2a_2^2 + \cdots)$ therefore represents what may be called the weight of the graduation formula or its effect in reducing individual errors. No general relation can, however, be deduced between the summations involved and the weight of the resulting formula except that it may be seen that the longer the interval over which the summations extend the greater will be the weight and that successive summations have a progressively decreasing effect.

47. In investigating the effect on the magnitude of the third difference it is necessary to expand δ^3G in terms of E, since δ is

a symbol of differencing. But we have, where H designates the preliminary operation,

$$\begin{aligned}
\delta^3 G &= \delta^3 \frac{[p][q][r]}{pqr} H = \frac{1}{pqr} [\overline{p}][\overline{q}][\overline{r}] H \\
&= \frac{1}{pqr} (E^{p/2} - E^{-(p/2)})(E^{q/2} - E^{-(q/2)})(E^{r/2} - E^{-(r/2)}) H \\
&= \frac{1}{pqr} (E^{(p+q+r)/2} - E^{(p+q-r)/2} - E^{(q+r-p)/2} - E^{(r+p-q)/2} \\
&\quad + E^{-[(p+q-r)/2]} + E^{-[(q+r-p)/2]} + E^{-[(r+p-q)/2]} - E^{-[(p+q+r)/2]}) H.
\end{aligned}$$

If $p = q = r = n$ this becomes

$$\delta^3 G = \frac{1}{n^3} \{E^{3n/2} - 3E^{n/2} + 3E^{-(n/2)} - E^{-(3n/2)}\} H.$$

In this case if the range of terms included in H when expanded in terms of E does not exceed n, or if there are no two significant terms in H separated by an interval which is a multiple of n, that is, if there are no like terms to be brought together after multiplying out, the sum of the squares of the coefficients of the powers of E in $\delta^3 G$ will be exactly $20/n^6$ times the sum of the squares of the coefficients in H. For example, in Woolhouse's formula,

$$\begin{aligned}
H &= -3E + 7 - 3E^{-1} \\
\delta^3 G_w &= \tfrac{1}{125}(E^{7\frac{1}{2}} - 3E^{2\frac{1}{2}} + 3E^{-2\frac{1}{2}} - E^{-7\frac{1}{2}})(-3E + 7 - 3E^{-1}) \\
&= \tfrac{1}{125}\{-3E^{8\frac{1}{2}} + 7E^{7\frac{1}{2}} - 3E^{6\frac{1}{2}} + 9E^{3\frac{1}{2}} - 21E^{2\frac{1}{2}} + 9E^{1\frac{1}{2}} - 9E^{-1\frac{1}{2}} \\
&\quad + 21E^{-2\frac{1}{2}} - 9E^{-3\frac{1}{2}} + 3E^{-6\frac{1}{2}} - 7E^{-7\frac{1}{2}} + 3E^{-8\frac{1}{2}}\}.
\end{aligned}$$

The sum of the squares of the coefficients in this expansion is readily seen to be equal to

$$\frac{1}{125^2}(1 + 9 + 9 + 1)(3^2 + 7^2 + 3^2) = \frac{20}{125^2}(3^2 + 7^2 + 3^2).$$

As the sum of the squares of the coefficients in the expansion of the expression for the third difference of the original series is 20, the effect of a graduation on the smoothness of the series is usually measured by a smoothing coefficient which is computed by dividing the sum of the squares of the coefficients in the expansion of the expression for the third difference of the graduated series by 20.

and extracting the square root of the quotient. Thus the smoothing coefficient in Woolhouse's graduation is $\sqrt{67}/125$ or about 1/15. In Higham's formula the coefficient similarly is $\sqrt{5}/125$ or about 1/56. (21) (25)

48. It will be seen from the general expression for the third difference of the graduated series that in general the following relations hold.

(*a*) The smoothing coefficient will decrease as the product pqr increases.

(*b*) The smaller the sum of the squares of the coefficients in H the smaller will be the smoothing coefficient.

(*c*) The substitution of unequal summations for equal ones will tend to reduce the coefficient although this tendency will be somewhat checked by the fact that for a formula of given range the product pqr will be smaller for unequal summations than for equal.

It has been shown (see Transactions Actuarial Society, Vol. XVII, p. 43) that the minimum smoothing coefficient in a summation formula covering $2n+1$ terms and expressed in the general form $G=\alpha_0+\alpha_1\gamma_1+\alpha_2\gamma_2+\cdots\alpha_n\gamma_n$ is obtained when α_x takes the form $(h+jx^2)\{(n+1)^2-x^2\}\{(n+2)^2-x^2\}\{(n+3)^2-x^2\}$ and h and j are so determined that $\alpha_0+2\alpha_1+2\alpha_2+\cdots 2\alpha_n=1$ and $\alpha_1+4\alpha_2+\cdots n^2\alpha_n=0$. This reduces to simpler form when we put $m=n+2$ so that the number of terms covered by the formula is $2m-3$. It may then be shown that the general expression for α_x is

$$\alpha_x=\frac{315\{(m-1)^2-x^2\}\{m^2-x^2\}\{(m+1)^2-x^2\}\{(3m^2-16)-11x^2\}}{8m(m^2-1)(4m^2-1)(4m^2-9)(4m^2-25)}.$$

49. In accordance with the above principles it is found that Hardy's modification of Higham's formula reduced the smoothing coefficient to 1/95 by substituting unequal for equal summations. The smoothing coefficient in Karup's 19-term formula is 1/106 due to the small coefficients in H, and the coefficient in Spencer's 21-term formula is 1/160 due to all three causes. If still greater smoothing power is required than is obtained in Spencer's formula, we may extend the number of terms to 27 and use the following:

$$G=\frac{[5][7][11]}{385}\{[3]-\gamma_3\}.$$

This formula is due to Mr. Kenchington. It is easily applied, is correct to third differences and its smoothing coefficient is 1/326. (24)

50. The way in which these theoretical results are verified in actual experience may be tested by differencing the section of a table which we have graduated by Woolhouse's, Higham's and Spencer's formulas. We find that the sum of the absolute values, irrespective of sign, of the eight third differences of $10^5 q_x$, is for Woolhouse's formula 149, for Higham's 43 and for Spencer's 19. It will be noticed that this sum is not reduced by the more powerful formulas in the full proportion called for by the theory. This may be partly an accidental variation due to the small number of terms, but there is a natural explanation from the fact that there is a residual irregularity due to the dropping of fractions, which no formula would eliminate. The mean value of the third difference error arising from this source is in fact approximately unity, so that the probable value of the sum irrespective of sign of eight such differences would be approximately 8.

51. Let us return now to the effect of the graduation on V_x, the regular part of the series, and determine, for the different formulas we have mentioned, the error introduced in a fifth difference curve.

For Woolhouse's formula we have

$$G = \frac{[5]^3}{5^3}\{10 - 3[3]\} = (1 + D^2 + \tfrac{17}{60}D^4)^3(1 - 3D^2 - \tfrac{1}{4}D^4)$$
$$= (1 + 3D^2 + \tfrac{77}{20}D^4)(1 - 3D^2 - \tfrac{1}{4}D^4)$$
$$= 1 - 5{\cdot}4D^4.$$

For Higham's formula

$$G = (1 + 3D^2 + \tfrac{77}{20}D^4)(1 - 3D^2 - \tfrac{5}{4}D^4) = 1 - 6{\cdot}4D^4.$$

For Hardy's formula

$$G = (1 + \tfrac{5}{8}D^2 + \tfrac{41}{384}D^4)(1 + D^2 + \tfrac{17}{60}D^4)(1 + \tfrac{35}{24}D^2 + \tfrac{707}{1152}D^4) \times (1 - 3D^2 - \tfrac{5}{4}D^4)$$
$$= 1 + \tfrac{1}{12}D^2 - 6\tfrac{361}{720}D^4.$$

For Karup's formula

$$G = (1 + 3D^2 + \tfrac{77}{20}D^4)(1 - 3D^2 - \tfrac{53}{20}D^4) = 1 - 7{\cdot}8D^4.$$

For Spencer's 21-term formula

$$
\begin{aligned}
G &= (1 + D^2 + \tfrac{17}{60}D^4)^2(1 + 2D^2 + \tfrac{7}{6}D^4)(1 - 4D^2 - 3\tfrac{1}{3}D^4)\\
&= (1 + 4D^2 + 6\,\tfrac{11}{15}D^4)(1 - 4D^2 - 3\tfrac{1}{3}D^4)\\
&= 1 - 12{\cdot}6D^4.
\end{aligned}
$$

For Kenchington's 27-term formula

$$
\begin{aligned}
G &= (1 + D^2 + \tfrac{17}{60}D^4)(1 + 2D^2 + \tfrac{7}{6}D^4)(1 + 5D^2 + 7\tfrac{5}{12}D^4)\\
&\qquad \times (1 - 8D^2 - 6\tfrac{2}{3}D^4)\\
&= (1 + 8D^2 + 25\tfrac{13}{15}D^4)(1 - 8D^2 - 6\tfrac{2}{3}D^4)\\
&= 1 - 44{\cdot}8D^4.
\end{aligned}
$$

We thus see that in general with an increase in the smoothing power of a formula there is combined an increase in the error and in particular that in going from Spencer's 21-term formula to the 27-term formula, we obtain a reduction in the smoothing coefficient from 1/160 to 1/326 at the expense of an increase in the error from $12.6D^4$ to $44.8D^4$. With so large a coefficient as 44.8 the error becomes appreciable in the case of some functions to which the formula might be applied. (20) (21) (22) (23)

52. As in the case of graphic graduation we may largely reduce this error, either by using a regular series calculated by a mathematical formula as a basis and graduating either the difference or the ratio of the two series or by a transformation similar to that described in art. 17–19.

53. A difficulty met with in graduating a series by a summation formula arises from the fact that in order to determine any given graduated value it is necessary to have ungraduated values over a considerable interval on both sides of the required value. There is therefore an interval at each end of the series, for which the graduated values cannot be obtained by the formula. There is an apparent exception to this in the case of a frequency distribution in which the terms gradually diminish toward the ends, and finally vanish, but it must be remembered in such a case that the series is really infinitely extended, the ungraduated values beyond the limits being zero. This case also arises in applying such a formula

to the l_x or d_x column of a mortality table since the values beyond the limiting age are all zero. In other cases, however, such as when the values of q_x or m_x are being graduated, it is necessary to adopt some device to complete the table. This is more especially necessary at the older ages, but is sometimes necessary at the young ages also. Various devices have been adopted to accomplish this object. One consists in making a hypothetical extension of the table to the extent necessary to permit of its completion. In the case of the q_x or m_x column at the older ages this extension may consist of a series of numbers increasing in geometrical progression with a common ratio approaching $1 \cdot 1$. At the younger ages a large enough group of ages is sometimes taken to obtain a fair number of deaths and the series extended by assuming this rate to be constant for all younger ages. Another device is to take the last three graduated values given by the formula as a basis and to extend the table as a third difference series through these three values with a third difference so determined that the expected deaths for the ages in question will equal the actual. At the younger ages the series sometimes becomes very irregular on account of insufficient data before it absolutely disappears, and in that case it may be advisable to ignore even some of the graduated values that can theoretically be obtained and apply the above method over the larger interval.

This difficulty also arises in any endeavor to graduate by this method select or analyzed mortality tables accentuated in this case by the rapid changes in the values of $q_{[x]+t}$ for changes in t where t is small. A device which has been used with some success for this purpose is to make a preliminary adjustment of the first-year and ultimate mortality. Then assume

$$q_{[x-t]+t} = q_x - f(t)(q_x - q_{[x]}),$$

where q_x is the rate of mortality for attained age x in the ultimate table and $q_{[x]}$ is the first-year rate for the same age. Determine now values of $f(t)$ which will reproduce the aggregate deaths of the second, third, fourth and fifths years and substitute a smooth series of values, remembering that $f(0) = 1$ and that $f(t)$ vanishes when the ultimate stage is reached. Then for each attained age determine modified ungraduated values of q_x and $q_{[x]}$ from the

equations

$$\sum_{t=0}^{t=\omega} E_{[x-t]+t} \cdot q_{[x-t]+t} = \sum_{t=0}^{t=\omega} E_{[x-t]+t} \cdot q_x - \sum_{t=0}^{t=\omega} f(t) E_{[x-t]+t} \cdot (q_x - q_{[x]}) = \sum_{t=0}^{t=\omega} \theta_{[x-t]+t},$$

$$\sum_{t=0}^{t=\omega} f(t) E_{[x-t]+t} \cdot q_{[x-t]+t} = \sum_{t=0}^{t=\omega} f(t) E_{[x-t]+t} \cdot q_x - \sum_{t=0}^{t=\omega} f(t)^2 E_{[x-t]+t} \cdot (q_x - q_{[x]}) = \sum_{t=0}^{t=\omega} f(t) \theta_{[x-t]+t}.$$

The values of q_x and $q_{[x]}$ are then each graduated by a summation formula and the values of $q_{[x]+1}$, $q_{[x]+2}$, etc., filled in from the equation

$$q_{[x-t]+t} = q_x - f(t)(q_x - q_{[x]}).$$

The values of q_x should first be graduated and then for the purpose of extending $q_{[x]}$ at the older ages as a preliminary to graduation an average percentage of the ultimate rates based on a fairly broad group may be used.

54. As the Mortality Table given on page 5 is very irregular great smoothing power is desirable in any formula applied to it. Let us therefore use Mr. Kenchington's 27-term formula given in Art. 49. We find by grouping the experience up to age 63 inclusive that the actual deaths are 67.3 per cent. of the expected by the $O^{M(5)}$ table. We accordingly insert 67.3 per cent. of the $O^{M(5)}$ rates of mortality at all ages from 63 down to 42 inclusive in order to obtain graduated values down to age 55. The following table shows the process of graduating the values of q_x.

GRADUATION OF DATA BY SUMMATION

x.	(1) $10^3 q_x$.	(2) $[3](1)$.	(3) $\gamma_3(1)$.	(4) $(2)-(3)$.	(5) $[11](11)$.	(6) $[7](5)$.	(7) $[5](6)$.	(8) $\frac{1}{385.10^3}(7) = q'_x$.
42	7							
43	7							
44	8							
45	8	24	16	8				
46	8	25	17	8				
47	9	26	18	8				
48	9	28	19	9				
49	10	29	20	9				

GRADUATION OF DATA BY SUMMATION

x.	(1) $10^3 q_x$	(2) $[3](1)$.	(3) $\gamma_3(1)$.	(4) (2)–(3).	(5) $[11](11)$.	(6) $[7](5)$	(7) $[5](6)$.	(8) $\frac{1}{385.10^3}(7) = q'_x$
50	10	31	21	10	111			
51	11	33	22	11	118			
52	12	35	24	11	126			
53	12	37	25	12	134	928		
54	13	39	27	12	142	992		
55	14	42	29	13	151	1,096	5,525	.0144
56	15	45	30	15	146	1,209	5,903	.0153
57	16	48	32	16	175	1,300	6,185	.0161
58	17	51	35	16	222	1,306	6,369	.0165
59	18	54	37	17	239	1,274	6,484	.0168
60	19	58	40	18	225	1,280	6,598	.0171
61	21	62	57	5	148	1,324	6,882	.0179
62	22	67	27	40	119	1,414	7,429	.0193
63	24	86	28	58	152	1,590	8,221	.0214
64	40	73	44	29	219	1,821	9,269	.0241
65	9	58	60	– 2	312	2,072	10,588	.0275
66	9	41	105	– 64	415	2,372	12,180	.0316
67	23	70	84	– 14	456	2,733	14,110	.0366
68	38	142	93	49	399	3,182	16,272	.0423
69	81	163	80	83	419	3,751	18,526	.0481
70	44	209	99	110	513	4,234	20,907	.0543
71	84	199	78	121	668	4,626	23,324	.0606
72	71	231	185	46	881	5,114	25,639	.0666
73	76	187	204	– 17	898	5,599	27,953	.0726
74	40	220	142	78	848	6,066	30,463	.0791
75	104	304	181	123	887	6,548	33,338	.0866
76	160	322	169	153	904	7,136	36,764	.0955
77	58	328	179	149	980	7,989	40,834	.106
78	110	261	258	3	1,150	9,025	45,658	.119
79	93	342	343	– 1	1,469	10,136	51,315	.133
80	139	386	264	122	1,751	11,372	57,574	.150
81	154	476	349	127	1,884	12,793	64,135	.167
82	183	543	346	197	1,998	14,248	70,724	.184
83	206	628	412	216	2,140	15,586	76,828	.200
84	239	698	396	302	2,401	16,725	82,157	.213
85	253	765	405	360	2,605	17,476	86,227	.224
86	273	768	512	256	2,807	18,122	88,216	.229
87	242	737	470	267	2,890	18,318		
88	222	770	479	291	2,635	17,575		
89	306	759	495	264	2,644			
90	231	763	560	203	2,336			
91	226	679	355	324	1,658			
92	222	766	556	210				
93	318	673	731	– 58				
94	133	701	476	225				
95	250	883	889	– 6				
96	500	1,000	1,318	–318				
97	250	1,417						
98	667	1,917						
99	1,000							

COMPARISON OF ACTUAL AND EXPECTED DEATHS

x.	q_x.	Expected Deaths.	Actual Deaths.	Deviation.	Accumulated Deviation.
55	.0144	.0	0	.0	.0
56	.0153	.1	0	+ .1	+ .1
57	.0161	.2	0	+ .2	+ .3
58	.0165	.3	0	+ .3	+ .6
59	.0168	.5	1	− .5	+ .1
60	.0171	.8	1	− .2	− .1
61	.0179	1.0	3	− 2.0	− 2.1
62	.0193	1.4	2	− .6	− 2.7
63	.0214	1.8	0	+ 1.8	− .9
64	.0241	2.4	4	− 1.6	− 2.5
65	.0275	2.9	1	+ 1.9	− .6
66	.0316	3.8	1	+ 2.6	+ 2.0
67	.0366	4.7	3	+ 1.7	+ 3.7
68	.0423	5.6	5	+ .6	+ 4.3
69	.0481	6.5	11	− 4.5	− .2
70	.0543	7.3	6	+ 1.3	+ 1.1
71	.0606	8.7	12	− 3.3	− 2.2
72	.0666	9.3	10	− .7	− 2.9
73	.0726	10.5	11	− .5	− 3.4
74	.0791	11.8	6	+ 5.8	+ 2.4
75	.0866	13.3	16	− 2.7	− .3
76	.0955	14.3	24	− 9.7	−10.0
77	.106	14.7	8	+ 6.7	− 3.3
78	.119	17.3	16	+ 1.3	− 2.0
79	.133	18.6	13	+ 5.6	+ 3.6
80	.150	20.6	19	+ 1.6	+ 5.2
81	.167	22.7	21	+ 1.7	+ 6.9
82	.184	23.2	23	+ .2	+ 7.1
83	.200	25.2	26	− .8	+ 6.3
84	.213	23.2	26	− 2.8	+ 3.5
85	.224	20.4	23	− 2.6	+ .9
86	.233	17.9	21	− 3.1	− 2.2
87	.242	16.0	16	0.0	− 2.2
88	.249	13.4	12	+ 1.4	− .8
89	.256	12.5	15	− 2.5	− 3.3
90	.263	10.3	9	+ 1.3	− 2.0
91	.270	8.4	7	+ 1.4	− .6
92	.278	7.5	6	+ 1.5	+ .9
93	.287	6.3	7	− .7	+ .2
94	.298	4.5	2	+ 2.5	+ 2.7
95	.311	3.7	3	+ .7	+ 3.4
96	.326	2.6	4	− 1.4	+ 2.0
97	.344	1.4	1	+ .4	+ 2.4
98	.365	1.1	2	− .9	+ 1.5
99	.389	.4	1	− .6	+ .9
		398.9	398	+42.6 −41.7	+62.1 −44.3

55. This process gives graduated values of q_x up to age 86 inclusive only and the value for age 86 is considerably affected by the scanty and irregular data at the extreme old ages. We therefore ignore this value and taking the values of q_x for ages 83, 84 and 85 as a basis assume that the values for older ages form a third difference series, of which the general term is

$$q_{83+n} = .200 + .013n - .002\frac{n(n-1)}{2} + y\frac{n(n-1)(n-2)}{6}.$$

The value of y is then determined so as to make the expected deaths for ages 86 and over equal to the actual. The required value of y is found to be .0003952 and the rates of mortality from age 86 to age 99 are inserted on this basis. The table on page 41 shows the comparison of the actual with the expected deaths.

GRADUATION BY MATHEMATICAL FORMULA.

(a) *Frequency Distributions.*

56. If a variable may have any value within certain limits, the curve in which the ordinate y is proportional to the chance of the variable having the value x represented by the abscissa, is called the curve of frequency. For example, let the variable be the length of a human life which may have any value from zero to the limits of the mortality table. The chance of the duration falling between x and $x + dx$ is evidently $(l_x - l_{x+dx})/l_0$ or $l_x\mu_x d_x/l_0$ so that the equation of the frequency-curve in this case is $y = l_x\mu_x$.

Similarly, if two points be taken at random on a straight line, the curve of frequency of the distance of the furthest of the two from a specified end is represented by the straight line $y = x$ between the limits 0 and a where a is the length of the line.

57. Where in any two cases the variables are comparable quantities, such as the heights of individuals in two different nations or the durations of life in two different groups of individuals, it is convenient to have some short method of comparison between the two; some coefficient or factor which will indicate whether one curve or the other falls, on the whole, on the higher values of the variable, and if so by how much.

The quantity most frequently used for this purpose is the mean value of the variable obtained by multiplying each value by its

probability and summing. In other words, if m_1 be the mean value and a and b the limits of the curve $y = U_x$ we have

$$m_1 = \frac{\int_a^b yxdx}{\int_a^b ydx}.$$

It is evident from the above method that if in one curve the value of m_1 is greater than in the other, the former falls on the average on higher values of the variable. It is also evident that represented geometrically, m_1 is the abscissa of the center of position of the area included between the curve and the base.

Other functions which have been used for this purpose are the median and the mode. The median is that value of the variable which it is as likely as not to exceed. Its value h is determined from the equation $\int_a^h ydx = \int_h^b ydx$, where a is the lower and b the upper limit of the value of x. The mode is the value of the variable whose probability is the greatest. For instance: in the case where the curve is represented by the equation $y = f(t) = l_{x+t}\mu_{x+t}$, t being the variable, we see that the mean value of t is the complete expectation of life at age x, that the median value is what is known as the vie probable or equation of life, and that the mode corresponds to the most probable after-lifetime.

58. When the mean value of the variable has been determined, the next question is, "How closely do the values of the variable cluster about this mean value or how widely dispersed are they?" Several methods might be proposed of measuring the degree of dispersion, but the most natural one in connection with the mean is the mean square of the departure designated by μ_2 when the departure is measured from the mean. (1) (27) (28) (30) (31)

The mean square of the departure from any given value is known as the second moment about that value. The value of the second moment m_2 about any given origin can be readily expressed in terms of m_1 the mean value of the variable, and μ_2 as follows:

Designating the operation of taking the mean value by writing M in front of the expression, we have

$$\mu_2 = M(x - m_1)^2 = Mx^2 - 2m_1Mx + m_1^2 = m_2 - 2m_1^2 + m_1^2$$
$$= m_2 - m_1^2$$

or

$$m_2 = \mu_2 + m_1^2.$$

It is thus evident that the second moment about any other value of the variable is greater than that about the mean value. In other words, taking the mean value as point of reference makes the second moment a minimum. It thus appears that the second moment has a natural connection with the mean value of the variable. The mean absolute departure, departure in either direction being considered positive has a similar connection with the median.

Other measures in terms of μ_2 are sometimes substituted for it in order to express the dispersion as a linear magnitude. The measures most frequently so used in connection with frequency-curves in general are the standard deviation and the modulus. The standard deviation, commonly denoted by σ is a quantity whose square is equal to the mean square of departure. In other words $\sigma^2 = \mu_2$. The modulus, sometimes denoted by c, is determined by the equation $c^2 = 2\mu_2$ and the name is derived by analogy from the normal exponential frequency-curve whose equation is $y = ke^{-(x^2/c^2)}$ in the case of which curve $\mu_2 = c^2/2$.

59. Having determined the mean value of the variable and the degree of dispersion from that mean value, the next question is whether the various possible values of the variable are dispersed symmetrically about the mean value or whether the curve is heaped up on one side and drawn out on the other. And as the first moment necessarily vanishes and the second moment can give us no information on the subject because departures in both directions enter into it positively, we are forced to look to the third moment, or the mean value of the cube of the departure, for a criterion. It is evident that if the curve is symmetrical, each positive departure will be balanced by a corresponding negative one, and so the mean value of the cube will vanish. It is thus evident that a value of the third moment, other than zero, is an indication of a lack of symmetry. Denoting by μ_3 the third moment about the mean and by m_3 the corresponding moment about any other point taken as origin, we have

$$\begin{aligned} \mu_3 &= M(x - m_1)^3 = Mx^3 - 3m_1Mx^2 + 3m_1^2Mx - m_1^3 \\ &= m_3 - 3m_1m_2 + 2m_1^3 = m_3 - 3m_1\mu_2 - m_1^3 \end{aligned}$$

or

$$m_3 = \mu_3 + 3m_1\mu_2 + m_1^3.$$

Of course, the curve is not necessarily absolutely symmetrical if μ_3 vanishes, but any marked lack of symmetry would likely show itself in the value of μ_3. The value of μ_3 is usually taken as a measure of the skewness or lack of symmetry, being divided by c^3 in order that the measure may be always numerical, and the quotient being designated by j so that we have $j = \mu_3/c^3$. Another function entering into the theory of curves of frequency is the quotient of the square of the third moment by the cube of the second moment and denoted by β_1 so that we have $\beta_1 = \mu_3^2/\mu_2^3$. But we have $j^2 = \mu_3^2/c^6 = \mu_3^2/8\mu_2^3$, so that $\beta_1 = 8j^2$.

60. Similarly, further information in relation to the curve can be secured by determining the moments of higher order, but we shall in this investigation only take into account the fourth moment μ_4 or the mean value of the fourth power of the departure from the mean, and the quantity β_2 determined from the equation $\beta_2 = \mu_4/\mu_2^2$.

The value of m_4, the fourth moment about any other point as origin, may be readily seen to be connected with that of μ_4 by the following relation

$$\mu_4 = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$$

or

$$m_4 = \mu_4 + 4m_1\mu_3 + 6m_1^2\mu_2 + m_1^4.$$

61. When it is desired to graduate a frequency distribution by means of a mathematical formula, the constants in the formula are ordinarily so determined as to make the area of the frequency curve and the moments, or mean values of the various powers of the variable, so far as possible the same as in the ungraduated distribution. It is therefore necessary to determine the moments in the ungraduated series and also to determine the relation between the moments and the arbitrary constants in the substituted series.

62. In calculating the moments in an ungraduated series, we generally meet with the difficulty that the exact values of the various items are not given, but we have the total number of cases falling within certain limits. It is customary in such cases to calculate the moments on the assumption that the cases falling within each interval are concentrated at the middle of that interval

and to make a correction to allow for the actual distribution. This correction for the case where the curve of frequency has close contact with the base at both limits may be derived as follows:

Let U_x designate the value of the ordinate of the curve of frequency corresponding to the value x of the abscissa and let V_x represent the area intercepted between the ordinates corresponding to the values $x - \frac{1}{2}$ and $x + \frac{1}{2}$, so that V_x is equal to $\int_{-\frac{1}{2}}^{+\frac{1}{2}} U_{x+h}dh$. Then from Taylor's series for U_{x+h} we have

$$V_x = U_x + \frac{1}{24}\frac{d^2U_x}{dx^2} + \frac{1}{1920}\frac{d^4U_x}{dx^4} + \cdots,$$

whence

$$\Delta^2 V_{x-1} = \frac{d^2V_x}{dx^2} + \frac{1}{12}\frac{d^4V_x}{dx^4} + \cdots$$

$$= \frac{d^2U_x}{dx^2} + \frac{1}{8}\frac{d^4U_x}{dx^4} + \cdots$$

and

$$\Delta^4 V_{x-2} = \frac{d^4V_x}{dx^4} + \cdots = \frac{d^4U_x}{dx^4} + \cdots,$$

whence

$$U_x = V_x - \tfrac{1}{24}\Delta^2 V_{x-1} + \tfrac{3}{640}\Delta^4 V_{x-2} - \cdots.$$

Now $m_n = \int x^n U_x d_x = \Sigma x^n U_x$, if $x^n U_x$ and its derived functions vanish at the limits of integration, whence

$$m_n = \Sigma x^n (V_x - \tfrac{1}{24}\Delta^2 V_{x-1} + \tfrac{3}{640}\Delta^4 V_{x-2} - \cdots).$$

But generally

$$\Sigma x^n \Delta^{2m} V_{x-m} = \Sigma\Sigma \frac{\underline{|2m}}{\underline{|m + x - y}\;\underline{|m + y - x}}(-1)^{m+x-y} x^n V_y$$

$$= \Sigma V_y \Delta^{2m}(y - m)^n.$$

Therefore

$$m_n = \Sigma V_x\{x^n - \tfrac{1}{24}\Delta^2(x-1)^n + \tfrac{3}{640}\Delta^4(x-2)^n - \cdots\}.$$

But

$$\Delta^2(x-1)^n = n(n-1)x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{12}x^{n-4} + \cdots,$$

and

$$\Delta^4(x-2)^n = n(n-1)(n-2)(n-3)x^{n-4} + \cdots.$$

Therefore

$$m_n = \Sigma V_x\Big(x^n - \frac{n(n-1)}{24}x^{n-2} + \frac{7n(n-1)(n-2)(n-3)}{5760}x^{n-4} - \cdots\Big)$$

$$= m'_n - \frac{n(n-1)}{24}m'_{n-2} + \frac{7n(n-1)(n-2)(n-3)}{5760}m'_{n-4} - \cdots,$$

where uncorrected moments are designated by accented letters. Putting n successively equal to 1, 2, 3 and 4 we have

$$m_1 = m_1',$$
$$m_2 = m_2' - \tfrac{1}{12},$$
$$m_3 = m_3' - \tfrac{1}{4}m_1',$$
$$m_4 = m_4' - \tfrac{1}{2}m_2' + \tfrac{7}{240},$$

or if the mean value is taken as origin

$$\mu_2 = \mu_2' - \tfrac{1}{12},$$
$$\mu_3 = \mu_3',$$
$$\mu_4 = \mu_4' - \tfrac{1}{2}\mu_2' + \tfrac{7}{240} = \mu_4' - \tfrac{1}{2}\mu_2 - \tfrac{1}{80}.$$

63. This investigation shows that we may either calculate the moments from the totals of the groups and then apply the adjustment afterwards, or we may calculate the true central ordinates by the formula

$$U_x = V_x - \tfrac{1}{24}\Delta^2 V_{x-1} + \tfrac{3}{640}\Delta^4 V_{x-2} - \text{etc},$$

and then, from these corrected ordinates, calculate the moments, which will then not require any further adjustment.

The latter course has an advantage, where a series determined by a mathematical formula is being fitted to the observed series, in that ordinates are more easily calculated from the formula than areas, so that the calculation of the true central ordinates in the observed series facilitates a comparison of the two. They also give, sometimes, a better idea of the nature of the law of the series than do the areas.

The moments may be computed directly by selecting any value of x as origin and calculating the successive values of $x^n U_x$ or

$x^n V_x$ and summing. Where a series of moments is to be computed, however, the work may be abbreviated by a summation process as follows:

Suppose that ω is the highest value of x which occurs and Let

$${}^{\text{i}}U_h = \sum_h^\omega U_x,$$

$${}^{\text{ii}}U_h = \sum_h^\omega {}^{\text{i}}U_x = \sum_h^\omega U_x \sum_{h}^{x} = \sum_h^\omega (x - h + 1)U_x,$$

$${}^{\text{iii}}U_h = \sum_h^\omega {}^{\text{ii}}U_x = \sum_h^\omega U_x \sum_h^x (x - h + 1)$$

$$= \sum_h^\omega \frac{(x - h + 1)(x - h + 2)}{2} U_x,$$

$${}^{\text{iv}}U_h = \sum_h^\omega {}^{\text{iii}}U_x = \sum_h^\omega U_x \sum_h^x \frac{(x - h + 1)(x - h + 2)}{2}$$

$$= \sum_h^\omega \frac{(x - h + 1)(x - h + 2)(x - h + 3)}{6} U_x,$$

$${}^{\text{v}}U_h = \sum_h^\omega {}^{\text{iv}}U_x$$

$$= \sum_h^\omega \frac{(x - h + 1)(x - h + 2)(x - h + 3)(x - h + 4)}{24} U_x.$$

The transformation is effected in each case by expanding and then collecting terms involving the same U_x. Whence

$${}^{\text{i}}U_1 = \sum_1^\omega U_x,$$

$${}^{\text{ii}}U_1 = \sum_1^\omega xU_x = m_1,$$

$$\tfrac{1}{2}({}^{\text{iii}}U_1 + {}^{\text{iii}}U_2) = \sum_1^\omega \frac{x^2}{2} U_x = \tfrac{1}{2}m_2,$$

$${}^{\text{iv}}U_2 = \sum_1^\omega \frac{x(x^2 - 1)}{6} U_x = \tfrac{1}{6}(m_3 - m_1),$$

$$\tfrac{1}{2}({}^{\text{v}}U_2 + {}^{\text{v}}U_3) = \sum_1^\omega \frac{x^2(x^2 - 1)}{24} U_x = \tfrac{1}{24}(m_4 - m_2). \qquad (1)$$

64. It is usually an economy of labor to divide the series into two at some central point and apply the summation separately

to the two sides with the central point as origin. In combining the two care must be taken regarding the signs, the summations giving odd moments being subtracted and those giving even moments added. The following is an example of the working of this method. The numbers included in brackets represent the final quantities entering into the formulas of Art. 63.

x.	V_x.	$\Delta^2 V_{x-1}$.	U_x.	${}^{\mathrm{i}}U_x$.	${}^{\mathrm{ii}}U_x$.	${}^{\mathrm{iii}}U_x$.	${}^{\mathrm{iv}}U_x$.	${}^{\mathrm{v}}U_x$.
−5	0	17	−1	11	11	11	11	
−4	17	121	12					
−3	155	156	148	159	170	181	192	203
−2	449	−215	458	617	787	968	1,160	(783)
−1	528	− 85	532	1,149	1,936	(1,936)		
0	522	− 42	524	(2,919)	(839)	(6,134.5)	(3,673)	(6,222.5)
1	474	− 71	477	1,246	2,775	(4,198.5)		
2	355	− 35	356	769	1,529	2,811	4,833	(5,439.5)
3	201	79	198	413	760	1,282	2,022	3,023
4	126	− 2	126	215	347	522	740	
5	49	71	46	89	132	175	218	
6	43	− 37	45	43	43	43	43	
7	0	43	−2					

The second difference correction only is used in this case and the negative values appearing for U_x are combined with the nearest positive value. The combined total in each column is entered opposite the value 0 of x, this and other adjusted values not forming part of the regular summations being enclosed in brackets.

We have then

$$m_1 = \frac{839}{2919} = .287;$$

$$m_2 = \frac{2 \times 6134.5}{2919} = 4.203; \ \mu_2 = 4.121;$$

$$(m_3 - m_1) = \frac{6 \times 3673}{2919} = 7.550; \ m_3 = 7.837; \ \mu_3 = 4.265;$$

$$(m_4 - m_2) = \frac{24 \times 6222.5}{2919} = 51.161; \ m_4 = 55.364; \ \mu_4 = 48.424.$$

65. In most frequency distributions it is observed that a single well-defined maximum appears and that as we depart from this value in either direction the frequency decreases and in most cases vanishes at the extreme limits. In graduating a frequency distribution of this kind by a mathematical formula, a general type

found frequently useful is that given by the differential equation

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{-(b+x)}{c+bx+ax^2},$$

where the mean value of x is taken as origin. The equations connecting the moments with the constants of the equation may be obtained as follows: We have

$$(b+x)ydx = -(c+bx+ax^2)dy.$$

Multiplying by x^{n-1} and integrating we have

$$\int_l^m (b+x)x^{n-1}ydx = -\int_l^m (c+bx+ax^2)x^{n-1}dy$$

$$= -[(c+bx+ax^2)x^{n-1}y]_l^m + \int_l^m \{(n-1)cx^{n-2}+nbx^{n-1}$$
$$+(n+1)ax^n\}ydx,$$

so that, if l and m are taken so that $(c+bx+ax^2)x^{n-1}y$ vanishes at both limits, we have

$$b\mu_{n-1}+\mu_n=(n-1)c\mu_{n-2}+nb\mu_{n-1}+(n+1)a\mu_n$$

or

$$\{1-(n+1)a\}\mu_n=(n-1)b\mu_{n-1}+(n-1)c\mu_{n-2}.$$

Whence putting n successively equal to 2, 3 and 4 and remembering that $\mu_0 = 1$ and $\mu_1 = 0$, we have

$$(1-3a)\mu_2 = c,$$
$$(1-4a)\mu_3 = 2b\mu_2,$$
$$(1-5a)\mu_4 = 3b\mu_3+3c\mu_2,$$

whence eliminating b and c we get

$$(1-5a)\mu_4=\frac{3(1-4a)}{2}\frac{\mu_3^2}{\mu_2}+3(1-3a)\mu_2^2$$

or, dividing through by $\mu_2^2/2$ and writing β_1 for μ_3^2/μ_2^3 and β_2 for μ_4/μ_2^2,

$$2(1-5a)\beta_2=3(1-4a)\beta_1+6(1-3a)$$

or

$$(2\beta_2-3\beta_1-6)=(10\beta_2-12\beta_1-18)a,$$

$$a=\frac{2\beta_2-3\beta_1-6}{10\beta_2-12\beta_1-18}=\frac{2(\beta_2+3)-3(\beta_1+4)}{10(\beta_2+3)-12(\beta_1+4)}=\frac{2-3\gamma}{10-12\gamma},$$

where

$$\gamma = \frac{\beta_1 + 4}{\beta_2 + 3},$$

$$b = \frac{1 - 4a}{2} \cdot \frac{\mu_3}{\mu_2} = \frac{1}{10 - 12\gamma} \frac{\mu_3}{\mu_2};$$

$$c = (1 - 3a)\mu_2 = \frac{4 - 3\gamma}{10 - 12\gamma} \cdot \mu_2.$$

The differential equation may therefore be written in the form

$$\frac{d \log y}{dx} = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{-\left\{\frac{\mu_3}{\mu_2} + (10 - 12\gamma)x\right\}}{(4 - 3\gamma)\mu_2 + \frac{\mu_3}{\mu_2}x + (2 - 3\gamma)x^2}$$

$$= \frac{-\left\{\frac{\mu_3}{\mu_2} + (2 + 4\delta)x\right\}}{(2 + \delta)\mu_2 + \frac{\mu_3}{\mu_2}x + \delta x^2},$$

where

$$\delta = 2 - 3\gamma = \frac{2\beta_2 - 3\beta_1 - 6}{\beta_2 + 3}.$$

It may be shown that for any real frequency distribution γ is positive and less than unity, so that δ lies between 2 and -1.

66. The form of the differential equation shows that in general $\log y$ becomes infinite either positively or negatively along with x and also for those values of x for which

$$(2 + \delta)\mu_2 + \frac{\mu_3}{\mu_2}x + \delta x^2 = 0.$$

There are therefore three general types of curve represented by this equation, according as the roots of the equation are real and of different signs, real and of the same sign or complex. In the first case the curve is limited in both directions, the limiting values being the two roots of the equation for which values y either vanishes or becomes infinite In the second case the curve is limited in one direction by the numerically least of the two roots and is unlimited in the other direction, while in the third case it is unlimited in both directions. Between the first and second cases we have the limiting case where one of the roots is infinite, and between the second and third we have the case of equal roots.

In both these cases the curve is limited in one direction. There are also the special cases where $\mu_3 = 0$ and the curve is symmetrical. Here we have the two general cases according as the roots are real or complex with the limiting case where they are infinite.

67. The criterion of the nature of the roots of the equation is

$$\frac{\frac{\mu_3^2}{\mu_2^2}}{4\delta(\delta+2)\mu_2} = \frac{\beta_1}{4\delta(\delta+2)}.$$

If this is negative, which is true when, and only when, δ is negative, the roots are real and of different sign. If it is positive and greater than unity they are real and of the same sign and if it is positive and less than unity, they are complex. We have therefore the eight types of curve determined by the relative values of β_1 and δ as follows:

Type.	Value of		Range of Curve.
	β_1.	$4\delta(\delta+2)$.	
I.........	$= 0$	< 0	Limited both directions
II.........	$= 0$	$= 0$	Unlimited
III.........	$= 0$	> 0	Unlimited
IV.........	> 0	< 0	Limited both directions
V.........	> 0	$= 0$	Limited one direction
VI.........	> 0	$> 0 < \beta_1$	Limited one direction
VII.........	> 0	$= \beta_1$	Limited one direction
VIII.........	> 0	$> \beta_1$	Unlimited

68. The function to be integrated in the case of each type is a standard form taken up in treatises on the integral calculus. We will therefore merely give the form of the equation of the curve of frequency in each case in its simplest form and the equations connecting the constants involved with the moments. The functions of the moments which enter into various types are as follows:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3};\quad \beta_2 = \frac{\mu_4}{\mu_2^2};\quad \gamma = \frac{\beta_1+4}{\beta_2+3};\quad \delta = 2 - 3\gamma = \frac{2\beta_2 - 3\beta_1 - 6}{\beta_2+3};$$

$$k_2 = \frac{\beta_1}{4\delta(\delta+2)}.$$

Type I:

$$\beta_1 = 0;\quad \delta < 0.$$

Equation

$$y = k(a^2 - x^2)^{(n/2)-1},$$

$$n = \frac{2(1+\delta)}{-\delta},$$

$$a^2 = \frac{\delta + 2}{-\delta}\mu_2 = (n+1)\mu_2, \quad a > 0$$

$$k = \frac{N}{a^{n-1}\sqrt{\pi}} \cdot \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)},$$

where N is the total number.

Type II:

$$\beta_1 = 0; \ \delta = 0$$

Equation

$$y = ke^{-x^2/c^2},$$

$$c^2 = 2\mu_2, \quad c > 0$$

$$k = \frac{2N}{c\sqrt{\pi}}.$$

Type III:

$$\beta_1 = 0; \quad \delta > 0$$

Equation

$$y = k(a^2 + x^2)^{-[(n/2)+1]},$$

$$n = \frac{2(1+\delta)}{\delta},$$

$$a^2 = \frac{\delta + 2}{\delta}\mu_2 = (n-1)\mu_2, \quad a > 0$$

$$k = \frac{Na^{n+1}}{\sqrt{\pi}} \cdot \frac{\Gamma[(n/2)+1]}{\Gamma[(n+1)/2]}.$$

Type IV:

$$\beta_1 > 0; \ \delta < 0.$$

Equation

$$y = k(a - x)^{np-1}(a + x)^{nq-1},$$

$$n = \frac{2(1+\delta)}{-\delta},$$

$$a^2 = (1 - k_2)(n+1)\mu_2, \quad a > 0$$

$$(q - p)a = \frac{\mu_3}{2\delta\mu_2}; \ p + q = 1,$$

$$k = \frac{N}{(2a)^{n-1}} \frac{\Gamma(n)}{\Gamma(np)\Gamma(nq)},$$

$$m_1 = \frac{\mu_3}{2\delta\mu_2} = (q - p)a.$$

Type V:

$$\beta_1 > 0;\ \delta = 0.$$

Equation

$$y = kx^{m-1}e^{-(x/a)},$$

$$m = \frac{4}{\beta_1};\ a = \frac{\mu_3}{2\mu_2},$$

$$k = \frac{N}{|a| \cdot a^{m-1}\,\Gamma(m)},$$

where $|a|$ is the absolute value of a, neglecting sign,

$$m_1 = ma.$$

Type VI:

$$0 < 4\delta(\delta + 2) < \beta_1.$$

Equation

$$y = k(x + a)^{-(nq+1)}(x - a)^{np-1},$$

$$n = \frac{2(1 + \delta)}{\delta},$$

$$a^2 = (k_2 - 1)(n - 1)\mu_2, \quad a > 0$$

$$(p + q)a = \frac{\mu_3}{2\delta\mu_2};\ q - p = 1,$$

$$k = N(2a)^{n+1}\frac{\Gamma(nq + 1)}{\Gamma(np)\Gamma(n + 1)},$$

$$m_1 = \frac{\mu_3}{2\delta\mu_2} = (p + q)a.$$

Type VII:

$$\beta_1 > 0;\ 4\delta(\delta + 2) = \beta_1.$$

Equation

$$y = kx^{-(n+2)}e^{-(a/x)},$$

$$n = \frac{2(1 + \delta)}{\delta} \quad \text{or} \quad \left(\frac{n}{n - 2}\right)^2 = 1 + \frac{\beta_1}{4},$$

$$a = \frac{1 + \delta}{\delta^2} \cdot \frac{\mu_3}{\mu_2} = \frac{n(n - 2)}{4} \cdot \frac{\mu_3}{\mu_2},$$

$$k = |a| \cdot a^n/\Gamma(n + 1),$$

$$m_1 = \frac{\mu_3}{2\delta\mu_2} = \frac{n - 2}{4} \cdot \frac{\mu_3}{\mu_2} = \frac{a}{n}.$$

Type VIII:

$$0 < \beta_1 < 4\delta(\delta + 2).$$

Equation

$$y = k(a^2 + x^2)^{-(n/2+1)}e^{\nu \tan^{-1}(x/a)},$$

$$n = \frac{2(1+\delta)}{\delta},$$

$$a^2 = (1 - k_2)(n-1)\mu_2, \quad a > 0$$

$$\nu a = \frac{1+\delta}{\delta^2}\frac{\mu_3}{\mu_2} = \frac{n(n-2)}{4}\frac{\mu_3}{\mu_2},$$

$$k = Na^{n+1} \Big/ \int_{-\pi/2}^{+\pi/2} \cos^n \theta e^{-\nu\theta} d\theta,$$

$$m_1 = \frac{\mu_3}{2\delta\mu_2} = \frac{n-2}{4}\frac{\mu_3}{\mu_2} = \frac{\nu a}{n}.$$

69. To illustrate the application of this formula take the frequency distribution for which the moments have already been worked out. We have

$$\mu_2 = 4.121; \quad \mu_3 = 4.265; \quad \mu_4 = 48.424,$$

whence

$$\beta_1 = .26; \quad \beta_2 = 2.85; \quad \gamma = .73; \quad \delta = -.19.$$

This shows that the curve is of type IV.

$$k_2 = -\frac{.26}{4 \times .19 \times 1.81} = -.19;$$

$$n = \frac{2 \times .81}{.19} = \frac{1.62}{.19} = 8.5;$$

$$a^2 = 1.19 \times 9.5 \times 4.121 = 46.59;$$

$$a = 6.9,$$

$$(p - q)a = \frac{4.265}{2 \times .19 \times 4.121} = 2.72;$$

$$p - q = .4;$$

$$p = .7; \; q = .3;$$

$$np = 5.95; \; nq = 2.55;$$

$$\log k = \log 2919 - 7.5 \log. 13.8 + \log \Gamma(8.5) - \log \Gamma(5.95) - \log \Gamma(2.55)$$

$$= 3.465234 - 7.5 \times 1.139879 + 4.147194 - 2.042232 - .139169$$

$$= \bar{4}.881934.$$

The origin must be taken so that the value of m_1 is -2.76, so that it must be at a point $.29 + 2.76$ or 3.05 above the original origin, and the range of the curve is therefore from -3.85 to $+9.95$ on the original scale. If we retransfer to the original origin the equation of the curve takes the form

$$y = k(9.95 - x)^{4.96}(3.85 + x)^{1.55}.$$

70. If we ignore the value of δ and consequently select type V we have

$$m = \frac{4}{.26} = 15.5,$$

$$a = \frac{4.265}{8.242} = .52,$$

$$\log k = \log 2919 - 15.5 \log .52 - \log \Gamma(15.5)$$

$$= 3.465234 + 15.5 \times \bar{1}.716003 - 11.524835 = \overline{12}.674434,$$

$$m_1 = 8.06,$$

and the equation of the curve referred to the original origin takes the form

$$y = k(x + 7.77)^{14.5}e^{-[(x+7.77)/.52]}.$$

71. Another system of curves sometimes used for graduating frequency distributions is that for which the general equation is

$$y = A_0\varphi(x) + A_3\varphi'''(x) + A_4\varphi^{\text{iv}}(x),$$

where

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-[(x-b)^2/2\sigma^2]}$$

and A_0, A_3, A_4, b and σ are arbitrary constants. If these constants are determined by the method of moments we have the following equations:

$$A_0 = N, \text{ the total number; } b = m_1; \ \sigma^2 = \mu_2; \ A_3 = -\frac{N\mu_3}{\underline{|3}};$$

$$A_4 = \frac{N\mu_4 - 3N\mu_2^2}{\underline{|4}}$$

also

$$\sigma^3\varphi'''(x) = -\varphi(x)\left\{3\frac{x-b}{\sigma} - \frac{(x-b)^3}{\sigma^3}\right\}$$

and

$$\sigma^4\varphi^{\text{iv}}(x) = \varphi(x)\left\{3 - 6\frac{(x-b)^2}{\sigma^2} + \frac{(x-b)^4}{\sigma^4}\right\}.$$

So that the equation may be written in terms of the moments as follows:

$$y = N\phi(x)\left[1 + \frac{\mu_3}{\sigma^3\lfloor 3}\left\{3\frac{x-b}{\sigma} - \frac{(x-b)^3}{\sigma^3}\right\} + \frac{\beta_2 - 3}{\lfloor 4} \times\left\{3 - 6\frac{(x-b)^2}{\sigma^2} + \frac{(x-b)^4}{\sigma^4}\right\}\right].$$

In applying this equation, however, to a curve of very marked skewness the formula fails through the appearance of negative ordinates of significant value which are inadmissible as representing a frequency distribution. It is only applicable therefore where the curve approximates to the normal and the constants A_3 and A_4 represent relatively small corrections. (3)

(b) *Mortality Tables.*

72. The general law which is most frequently used in the graduation of mortality tables is the one devised by Makeham as a modification of the Gompertz formula. According to this law, the force of mortality at any given age x may be expressed by the function $A + Bc^x$. Another way of expressing this law is by the equation $l_x = ks^xg^{c^x}$. In the expression for the force of motality there are three arbitrary constants A, B and c, while in the expression for the value of l_x there are four but the fourth constant merely determines the radix of the mortality table and does not affect in any way the rate of mortality, nor the monetary values depending thereon. The arbitrary constants may be determined directly from the original data of the exposed to risk and deaths or a mortality table may have been constructed from the unadjusted death rates and the original data may not be available or it may be desired to graduate the rough table without direct reference to the original data. (34)

73. Where a graduated table is to be constructed directly from the original data, it is usually most convenient to so determine the exposed to risk as to give central death rates or force of mortality, rather than rates of mortality. When this is done the most difficult part of the problem is the determination of the value of c.

If the value of c is known and if we designate by L_x the number exposed to risk in the middle of the year of age x and by θ_x, the number of deaths occurring in the same year of age, we may determine A and B from the simultaneous simple equations

$$A\Sigma L_x + B\Sigma L_x c^{x+\frac{1}{2}} = \Sigma\theta_x,$$
$$A\Sigma xL_x + B\Sigma L_x xc^{x+\frac{1}{2}} = \Sigma x\theta_x.$$

These equations express the conditions that the total number of actual deaths is equal to the expected and that the mean age is the same in the two groups. The problem under this method of graduation, therefore, substantially resolves itself into the determination of the value of c.

74. One method which has been suggested for determining this value is by fitting a frequency curve of type V to the exposed to risk, and recomputing the deaths at each age by multiplying the adjusted value of the exposed by the unadjusted force of mortality. In this case the equation of the curve is $y = kx^{m-1}e^{-(x/a)}$, where x is not necessarily measured from age 0, but the origin is determined according to the methods already described (Art. 68). In this case let E_0, E_1, etc., represent the successive moments for the exposures around the origin, E_0', E_1', etc., the similar moments of the exposures multiplied by c^x and θ_0, θ_1, etc., the similar moments of the recomputed deaths. If then we put λ for $\log c$, the force of mortality may be written in the form $A + Be^{\lambda x}$ and the equation for the adjusted curve of deaths will be $y = Akx^{m-1}e^{-(x/a)} + Bkx^{m-1}e^{x[\lambda-(1/a)]}$, where the second term is of the same form as the first except that $(1/a) - \lambda$ is substituted for $1/a$. Then we have by the well-known properties of the gamma integral

$$\theta_0 = AE_0 + BE_0'$$

or

$$\frac{\theta_0}{E_0} = A + B\frac{E_0'}{E_0},$$

$$\theta_1 = AE_1 + BE_1' = AmaE_0 + B\frac{ma}{1 - a\lambda}E_0'$$

or

$$\frac{\theta_1}{E_1} = A + B\frac{1}{1 - a\lambda}\frac{E_0'}{E_0},$$

$$\theta_2 = AE_2 + BE_2' = Am(m + 1)a^2E_0 + B\frac{m(m + 1)a^2}{(1 - a\lambda)^2}E_0'$$

or

$$\frac{\theta_2}{E_2} = A + B\frac{1}{(1 - a\lambda)^2}\frac{E_0'}{E_0};$$

$$\frac{\theta_1}{E_1} - \frac{\theta_0}{E_0} = \frac{a\lambda}{1 - a\lambda}B\frac{E_0'}{E_0};\ \frac{\theta_2}{E_2} - \frac{\theta_1}{E_1} = \frac{a\lambda}{(1 - a\lambda)^2}B\frac{E_0'}{E_0};$$

$$(1 - a\lambda) = \frac{\dfrac{\theta_1}{E_1} - \dfrac{\theta_0}{E_0}}{\dfrac{\theta_2}{E_2} - \dfrac{\theta_1}{E_1}},$$

from which λ may be determined.

75. This method is open to the objection that ordinarily the exposed to risk at the extreme ages will be overstated by the frequency curve, and significant values may be obtained for ages which are not represented among the actual observations, and it will therefore be necessary to use hypothetical death rates at these ages. A curve of type IV might be used instead, but in that case it is necessary to take into account the third moments of the recomputed deaths and the weight of the determination of the value of c is thus very much reduced. (1)

76. Another method is to combine the exposed to risk into a few large groups and to fit a binomial series to the series so obtained. Let us suppose that in the substituted series the exposed to risk at age x is the coefficient of y^{x-l} in the expansion of $N(p + qy^a)^n$ where $p + q = 1$. Then we have

$$\mu_2 = na^2pq,$$
$$\mu_3 = na^3pq(p - q),$$
$$\mu_4 = a^4\{npq(1 - 6pq) + 3n^2p^2q^2\};$$
$$\beta_1 = \frac{(p - q)^2}{npq};\quad \beta_2 = 3 + \frac{1}{n}\left(\frac{1}{pq} - 6\right),$$

whence

$$n = \frac{2}{3 + \beta_1 - \beta_2},$$
$$a^2 = \mu_2(\beta_1 + 4/n)$$
$$a(p - q) = \mu_3/\mu_2$$
$$m_1 - l = nqa.$$

The quantities a and n, however, are for convenience ordinarily taken as integral so that these equations can only be approximately

satisfied. When the substituted series has, however, been determined it is necessary to multiply each term by the proper force of mortality, and the most convenient way of determining the proper force of mortality to use is by dividing the exposed to risk and the deaths into groups such that the central ordinate of each group will correspond to one of the ages for which the force of mortality is required. The central ordinate of each group can then be determined by the formula already given in Art. 62 and the force of mortality determined by making the proper division. By this method all of the data is brought into the calculation. If then we equate the actual number and the first and second moments taken about the point $x = l$ of the recomputed deaths to the corresponding functions of the expected we have the following equations for determining the value of c

$$\begin{aligned}
\theta_0 &= AE_0 + B(p + qc^a)^n E_0, \\
\theta_1 &= AnaqE_0 + Bnaqc^a(p + qc^a)^{n-1}E_0, \\
\theta_2 - a\theta_1 &= An(n-1)a^2q^2E_0 \\
&\qquad\qquad + Bn(n-1)a^2q^2c^{2a}(p + qc^a)^{n-2}E_0, \\
\frac{\theta_0}{E_0} &= A + B(p + qc^a)^n, \\
\frac{\theta_1}{E_1} &= A + Bc^a(p + qc^a)^{n-1}, \\
\frac{\theta_2 - a\theta_1}{E_2 - aE_1} &= A + Bc^{2a}(p + qc^a)^{n-2}, \\
\frac{\theta_1}{E_1} - \frac{\theta_0}{E_0} &= Bp(c^a - 1)(p + qc^a)^{n-1}, \\
\frac{\theta_2 - a\theta_1}{E_2 - aE_1} - \frac{\theta_1}{E_1} &= Bpc^a(c^a - 1)(p + qc^a)^{n-2}, \\
\frac{\dfrac{\theta_1}{E_1} - \dfrac{\theta_0}{E_0}}{\dfrac{\theta_2 - a\theta_1}{E_2 - aE_1} - \dfrac{\theta_1}{E_1}} &= \frac{p + qc^a}{c^a} = pc^{-a} + q.
\end{aligned}$$

77. Another method of determining the value of c direct from the original data suggested by Mr. Hardy is to arrange the exposed to risk and the deaths in quinquennial groups, calculating the central ordinates for each group and the corresponding force of mortality by the method just outlined. This gives us a series of

values of μ_x proceeding by intervals of 5 years. The values at the extreme ages are rejected and beginning at say $32\frac{1}{2}$ the six successive values covering ages $32\frac{1}{2}$ to $57\frac{1}{2}$ are combined after weighting them respectively by the factors 1, 3, 5, 5, 3, 1. The six values for ages $47\frac{1}{2}$ to $72\frac{1}{2}$ are similarly combined and also those for ages $62\frac{1}{2}$ to $87\frac{1}{2}$. We have then the following equations.

$$\begin{aligned}
\Sigma_1 &= 18A + Bc^{32\frac{1}{2}}(1 + 3c^5 + 5c^{10} + 5c^{15} + 3c^{20} + c^{25}) \\
&= 18A + Bc^{32\frac{1}{2}}f(c), \\
\Sigma_2 &= 18A + Bc^{47\frac{1}{2}}f(c), \\
\Sigma_3 &= 18A + Bc^{62\frac{1}{2}}f(c), \\
\Sigma_2 - \Sigma_1 &= Bc^{32\frac{1}{2}}(c^{15} - 1)f(c), \\
\Sigma_3 - \Sigma_2 &= Bc^{47\frac{1}{2}}(c^{15} - 1)f(c), \\
\frac{\Sigma_3 - \Sigma_2}{\Sigma_2 - \Sigma_1} &= c^{15}.
\end{aligned}$$

78. It has been found by experience that the value of $\log c$ in practically every case falls between .035 and .045 and satisfactory results can sometimes be obtained with much less labor than by either of the above methods by using trial values of $\log c$ beginning with .04. When this method is followed the graduated table resulting from each value of c is tested in the usual method and the table which gives the most satisfactory results is selected.

79. It will be found, however, that, unless the value of $\log c$ which is adopted agrees closely with the data, if the values of A and B are derived according to the method described in Art. 73 and the corresponding rates of mortality computed and applied to the exposed to risk the total number of deaths and the first moment will not be reproduced. If, therefore, it is desired to obtain a mortality table based on a selected value of c, not necessarily agreeing with the data, which will reproduce the total number of deaths and the first moment a different process is required. We have

$$\begin{aligned}
\operatorname{colog}_{10} p_x &= \alpha + \beta c^x = \alpha + c^{x-m}; \\
\therefore\quad p_x &= 10^{-\alpha}p'_{x-a} = rp'_{x-a},
\end{aligned}$$

where p_x' is so determined that $\operatorname{colog}_{10} p_x' = c^{x-m+a}$. We therefore select an arbitrary value of $m - a$ and the problem reduces to the determination of the values of r and a. The method of

Art. 73 will give an approximate value of a and it is then easy by a method of trial and error to determine the value of a for which

$$\frac{\Sigma x E_x p'_{x-a}}{\Sigma E_x p'_{x-a}} = \frac{\Sigma x(E_x - \theta_x)}{\Sigma(E_x - \theta_x)}.$$

We then determine r from the equation $r\Sigma E_x p'_{x-a} = \Sigma(E_x - \theta_x)$.

From these two equations it follows that where $p_x = rp'_{x-a}$ we have

$$\Sigma(E_x q_x - \theta_x) = 0,$$
$$\Sigma x(E_x q_x - \theta_x) = 0.$$

In other words the sum of the deviations and of the accumulated deviations will vanish.

80. Applying the method of Art. 73 to the mortality experience given on page 5 we obtain $\text{Colog}_{10}(p_x) = .00193 + .00003737c^x$ on the assumption that $\log_{10} c = .04$. This gives us 110.7 as an approximate value of m. Taking therefore 100 as the arbitrary value of $m - a$ and using 10 and 11 as trial values, we have

$$\frac{\Sigma(100 - x)(E_x - \theta_x)}{\Sigma(E_x - \theta_x)} = \frac{82{,}938}{3{,}220} = 25.7571,$$
$$\frac{\Sigma(100 - x)E_x p'_{x-10}}{\Sigma E_x p'_{x-10}} = \frac{82{,}868.06}{3{,}214.21} = 25.7818,$$
$$\frac{\Sigma(100 - x)E_x p'_{x-11}}{\Sigma E_x p'_{x-11}} = \frac{83{,}501.88}{3{,}246.59} = 25.7199.$$

We take therefore 10.4 as the value of a and we have

$$r = \frac{\Sigma(E_x - \theta_x)}{\Sigma E_x p'_{x-10.4}} = \frac{3{,}220}{3{,}227.16},$$

$$\text{Colog}_{10} r = .00096.$$

Hence we have finally

$$\text{Colog}_{10} p_x = .00096 + 10^{.04(x-110.4)}.$$

The table on page 63 shows the resulting rates of mortality and the comparison of the actual with the expected claims.

81. The value of c determined by any of these methods may, however, be considered as approximate only and in this case, if the true value of c be designated by $c + \delta c$, the expression for μ_x

GRADUATION OF DATA BY MAKEHAM'S FORMULA

x.	q_x.	Expected Deaths.	Actual Deaths.	Deviations.	Accumulated Deviation.
55	.0161	.0	0	.0	.0
56	.0174	.1	0	+ .1	+ .1
57	.0189	.2	0	+ .2	+ .3
58	.0205	.4	0	+ .4	+ .7
59	.0222	.7	1	− .3	+ .4
60	.0241	1.2	1	+ .2	+ .6
61	.0262	1.5	3	− 1.5	− .9
62	.0285	2.1	2	+ .1	− .8
63	.0310	2.6	0	+ 2.6	+ 1.8
64	.0337	3.4	4	− .6	+ 1.2
65	.0367	3.9	1	+ 2.9	+ 4.1
66	.0400	4.6	1	+ 3.6	+ 7.7
67	.0435	5.6	3	+ 2.6	+ 10.3
68	.0474	6.3	5	+ 1.3	+ 11.6
69	.0517	7.0	11	− 4.0	+ 7.6
70	.0563	7.6	6	+ 1.6	+ 9.2
71	.0614	8.8	12	− 3.2	+ 6.0
72	.0669	9.4	10	− .6	+ 5.4
73	.0729	10.5	11	− .5	+ 4.9
74	.0794	11.8	6	+ 5.8	+ 10.7
75	.0866	13.3	16	− 2.7	+ 8.0
76	.0944	14.2	24	− 9.8	− 1.8
77	.103	14.3	8	+ 6.3	+ 4.5
78	.112	16.2	16	+ .2	+ 4.7
79	.122	17.1	13	+ 4.1	+ 8.8
80	.133	18.2	19	− .8	+ 8.0
81	.144	19.6	21	− 1.4	+ 6.6
82	.157	19.8	23	− 3.2	+ 3.4
83	.170	21.4	26	− 4.6	− 1.2
84	.185	20.2	26	− 5.8	− 7.0
85	.201	18.3	23	− 4.7	− 11.7
86	.218	16.8	21	− 4.2	− 15.9
87	.236	15.6	16	− .4	− 16.3
88	.255	13.8	12	+ 1.8	− 14.5
89	.276	13.5	15	− 1.5	− 16.0
90	.298	11.6	9	+ 2.6	− 13.4
91	.321	10.0	7	+ 3.0	− 10.4
92	.346	9.3	6	+ 3.3	− 7.1
93	.372	8.2	7	+ 1.2	− 5.9
94	.400	6.0	2	+ 4.0	− 1.9
95	.429	5.1	3	+ 2.1	+ .2
96	.459	3.7	4	− .3	− .1
97	.490	2.0	1	+ 1.0	+ .9
98	.521	1.6	2	− .4	+ .5
99	.554	.6	1	− .4	+ .1
		398.1	398	+51.0 −50.9	+128.3 −124.9

becomes approximately $A + Bc^x + Bxc^{x-1}\delta c$, which may be written in the form $A + Bc^x + Dxc^x$, where $D = B\delta c/c$. If we then equate the total number and the first and second moments of the actual deaths with the corresponding functions of the expected, we have the following equations for determining A, B and D

$$\begin{aligned} AE_0 + BE_0' + DE_1' &= \theta_0, \\ AE_1 + BE_1' + DE_2' &= \theta_1, \\ AE_2 + BE_2' + DE_3' &= \theta_2. \end{aligned}$$

In this case the moments are all taken about the value zero of x.

82. When a mortality table is to be graduated without direct reference to the original data the simplest way in which the constants can be determined is by using four equidistant values of $\log l_x$ in which case we have the following equations for determining the constants:

$$\begin{aligned} \log l_x &= \log k + x \log s + c^x \log g, \\ \log l_{x+t} &= \log k + (x + t) \log s + c^{x+t} \log g, \\ \log l_{x+2t} &= \log k + (x + 2t) \log s + c^{x+2t} \log g, \\ \log l_{x+3t} &= \log k + (x + 3t) \log s + c^{x+3t} \log g. \end{aligned}$$

Denoting then the difference over an interval t by the symbol Δ, we have

$$\begin{aligned} \Delta \log l_x &= t \log s + c^x(c^t - 1) \log g, \\ \Delta^2 \log l_x &= c^x(c^t - 1)^2 \log g, \end{aligned}$$

whence

$$\begin{aligned} c^t &= \frac{\Delta^2 \log l_{x+t}}{\Delta^2 \log l_x}, \\ \log g &= \frac{\Delta^2 \log l_x}{c^x(c^t - 1)^2}, \\ t \log s &= \Delta \log l_x - c^x(c^t - 1) \log g = \Delta \log l_x - \frac{\Delta^2 \log l_x}{c^t - 1}, \\ \log k &= \log l_x - x \log s - c^x \log g. \end{aligned}$$

83. It will be seen that by this method the integral of μ_x over each interval is made the same in the graduated table as in the ungraduated and that the experience not included in the three intervals is ignored. In view of the somewhat excessive importance placed on particular points of division by this method and of the varying results which may be obtained by the selection

of different ages and intervals, this method is usually modified by using instead of a single value of $\log l_x$ the sum of the values for a group of adjacent ages. In this case we have the following equations for determining the constants:

$$\sum_x^{x+n-1} \log l_x = n \log k + n\left(x + \frac{n-1}{2}\right) \log s + c^x \frac{c^n - 1}{c - 1} \log g,$$

$$\Delta\Sigma \log l_x = nt \log s + \frac{c^x(c^t - 1)(c^n - 1)}{c - 1} \log g,$$

$$\Delta^2\Sigma \log l_x = \frac{c^x(c^t - 1)^2(c^n - 1)}{c - 1} \log g,$$

whence

$$c^t = \frac{\Delta^2\Sigma \log l_{x+t}}{\Delta^2\Sigma \log l_x},$$

$$\log g = \frac{(c - 1)\Delta^2\Sigma \log l_x}{c^x(c^t - 1)^2(c^n - 1)},$$

$$nt \log s = \Delta\Sigma \log l_x - \frac{c^x(c^t - 1)(c^n - 1)}{c - 1} \log g$$

$$= \Delta\Sigma \log l_x - \frac{\Delta^2\Sigma \log l_x}{c^t - 1},$$

$$n \log k = \Sigma \log l_x - \frac{n(2x + n - 1)}{2} \log s - \frac{c^x(c^n - 1)}{c - 1} \log g.$$

Where the number n of ages included in each group is the same as the interval t, this method becomes the one proposed by King and Hardy and described in the Institute Text-Book. (35) (36)

84. This modification minimizes but does not entirely eliminate the importance given to special points of division and Professor Karl Pearson has suggested the application of the methods of moments direct to the values of $\log l_x$, determining the constants so as to reproduce the integral of $\log l_x$, within suitable limits, and its first, second and third moments. As, however, we have only the value of $\log l_x$ for integral values of x, this method in its original form requires the use of a formula of approximate integration. The work, however, can be simplified without interfering with the simplicity of the resulting equations by using summation instead of integration. Under this method we have the following equations for determining the constants:

$$\log l_{a+x} = \log k + (a + x) \log s + c^{a+x} \log g$$
$$= (\log k + a \log s) + x \log s + c^x \cdot c^a \log g$$
$$= \log k' + x \log s + c^x \log g',$$

$${}_1S_x = \sum_0^{x-1} \log l_{a+x} = x \log k' + \frac{x^{(2)}}{2} \log s + \frac{c^x - 1}{c - 1} \log g',$$

$${}_2S_x = \sum_0^{x-1} {}_1S_x = \frac{x^{(2)}}{2} \log k' + \frac{x^{(3)}}{\lfloor 3} \log s$$
$$+ \left\{ \frac{c^x - 1}{(c-1)^2} - \frac{x}{c-1} \right\} \log g',$$

$${}_3S_x = \sum_0^{x-1} {}_2S_x = \frac{x^{(3)}}{\lfloor 3} \log k' + \frac{x^{(4)}}{\lfloor 4} \log s$$
$$+ \left\{ \frac{c^x - 1}{(c-1)^3} - \frac{x}{(c-1)^2} - \frac{x^{(2)}}{2(c-1)} \right\} \log g',$$

$${}_4S_x = \sum_0^{x-1} {}_3S_x = \frac{x^{(4)}}{\lfloor 4} \log k' + \frac{x^{(5)}}{\lfloor 5} \log s$$
$$+ \left\{ \frac{c^x - 1}{(c-1)^4} - \frac{x}{(c-1)^3} - \frac{x^{(2)}}{2(c-1)^2} - \frac{x^{(3)}}{\lfloor 3\,(c-1)} \right\} \log g'$$

where, generally, $x^{(r)} = x(x-1) \cdots (x-r+1)$.

Putting then n for x in these last four equations, where n is the total range to be included in the summations, and eliminating $\log k'$, $\log s$ and $\log g'$, by multiplying the equations through by $2/n$, $\lfloor 3/n^{(2)}$, $\lfloor 4/n^{(3)}$ and $\lfloor 5/n^{(4)}$ respectively, differencing twice and taking the ratio of the two second differences we obtain an equation in c which may be solved by successive approximations to any required degree of accuracy. The other constants are then determined from simple equations. (37)

85. Another method of graduating mortality tables is to make use of a hypothetical table of exposed to risk conforming with a frequency distribution of such a type as to give manageable equations for determining the constants. The actual working of this method is the same as in the case where a graduated table is constructed direct from the original data by fitting a frequency curve to the exposed to risk, as described in article 74.

86. In the preceding discussion it has been assumed that the mortality is analyzed only according to attained age, and where select or analyzed tables are required some modification of the

method is usually necessary. This is usually done by making A and B functions of the duration, the value of c being taken as the same for all durations. For practical reasons, the functions A and B are usually assumed to become constant after some definite duration, which in some cases is fixed at five years and in some cases at ten years. The method usually followed is to set out the experience for each year of duration so far as it is intended to follow selection and to determine by some of the methods already described the corresponding values of A and B, the data for each year of duration being treated as representing a mortality table complete in itself. These values will, however, be somewhat irregular, so that they themselves require further graduation. In graduating tables of this kind it is usually better to work with colog p_x rather than μ_x, the general form of the two functions being identical. In selecting formulas for graduating these rough values, the following conditions should be satisfied: (1) A smooth junction between the curves representing the select and ultimate tables. (2) An agreement between the graduated and ungraduated values of colog p_x in year 0 as special importance is attached to the rate of mortality in the first year of insurance. (3) An agreement between the aggregate graduated and ungraduated values of these functions during the period between the date of entry and the ultimate table. Considerable experimenting will usually be necessary to determine a function complying with these conditions. The final form of the equation which was adopted for the $O^{[M]}$ experience was as follows:

$$\log_{10} l_{[x]+t} = \log_{10} l_{x+t} - f_t - \beta c^x \psi_t,$$

where

$$f_t = m(10 - t)^2 + m'(c')^t,$$

and

$$\psi_t = n(10 - t)^2. \quad (1)\ (40)\ (41)$$

87. When an analyzed mortality table is being constructed it may be desired to merge the analyzed tables into the ultimate at a duration somewhat shorter than that over which the effect of selection is known to extend. In this case a mortality table based on the average experience for all longer durations will not give correct values for the annuity or for the expectation of life at the point of junction, but will give values too high at the young ages

and too low at the older ages, and this error will affect the corresponding values at date of entry. If, therefore, the values at date of entry are more important than those for longer durations, it is necessary to so determine the constants as to reproduce those values as nearly as possible. The problem presented is therefore to graduate by Makeham's formula a series of annuity values or of values of the expectation of life. As the expectation of life is merely the annuity value for zero interest the annuity may be taken as the general case for discussion.

The same problem also arises where a mortality table has been in general use and has been adopted as a standard, and it is desired to regraduate it by Makeham's law for certain special purposes. In this case also it is important that the monetary values should be reproduced as nearly as possible.

88. In the following discussion we will assume that approximate values of the constants are known. These values may be derived by any of the methods already described which will apply to the particular case. The next step is then to assign weights to the various ungraduated values used as a basis. In the case of an analyzed table these values will ordinarily proceed by quinquennial or other intervals, each value being derived from the experience of a group of entry ages suitably corrected to reduce it to the central age of the group. If then n is the total number exposed to risk and nq the total number of deaths in the group upon which the annuity value is based the mean deviation, irrespective of sign in the number of deaths, is approximately $.8\sqrt{nq(1-q)}$, the ratio of which to the total number of deaths is $.8\sqrt{(1-q)/nq}$. If now from the approximate graduation we construct a table showing the change in the various values of a_x corresponding to a change of 1 per cent. in the mortality, the average deviation in the annuity values may be expressed approximately by the product of one hundred times this change and the above ratio. The annuity values may then be reduced to the same weight by multiplying them by numbers in proportion to the reciprocals of their average deviation. If a standard table is being regraduated weights may be assigned in proportion to the relative importance of the different sections of the table. (1)

89. Suppose now that the force of mortality may be expressed in the form $\mu_x = A + Bc^x = A' + h + B'c^{x+k}$, where A' and B' are approximate values of A and B, and let accented letters

designate values computed on the basis of these approximate values. Then we have approximately

$$a_x = a_x' + h\frac{da_x'}{dA'} + k\frac{da_x'}{dx}$$
$$= a_x' - h(Ia')_x + k\frac{da_x'}{dx}.$$

The values of h and k are then determined by the method of moments. If we designate the ungraduated values of a_x by a double accent and denote the weights by w_x the equations are

$$\Sigma w_x(a_x'' - a_x') = -h\Sigma w_x(Ia')_x + k\Sigma w_x\cdot\frac{da_x'}{dx},$$
$$\Sigma x w_x(a_x'' - a_x') = -h\Sigma x w_x(Ia')_x + k\Sigma x w_x\cdot\frac{da_x'}{dx}.$$

If the values of h and k so determined are considerable they should not be accepted as final but the constants so arrived at should be used as a basis for a further approximation.

90. In the foregoing it has been assumed that the value of c was known. Where it is to be determined we may assume trial values of c and graduate by the above method and then select the value giving the best graduation or we may suppose it subject to variation and adopt the following process.

Let

$$\mu_x = A + Bc^x = A + Be^{\lambda x}$$
$$= A' + h + B'e^{(\lambda'+l)(x+k)}$$

Then we have

$$\bar{a}_x = \bar{a}_x' + h\frac{d\bar{a}_x'}{dA'} + k\frac{d\bar{a}_x'}{dx} + l\frac{d\bar{a}_x'}{d\lambda'}.$$

And h, k and l may be determined by the method of moments, the second moments being brought in to furnish a third equation.

91. To obtain an expression for $d\bar{a}_x'/d\lambda$ let us designate the value of $\bar{a}_x$ calculated on the basis of the constants A, B and λ by $f(A, B, \lambda, x)$. Then we have

$$f\{(A + r\overline{A + \delta}),\ (1 + r)B,\ (1 + r)\lambda,\ x\}$$
$$= \frac{1}{1+r}f\{A, B, \lambda, (1 + r)x\}$$

for all values of r, when δ is the force of interest. Let then r become indefinitely small and we have

$$(A + \delta)\frac{d\bar{a}_x}{dA} + B\frac{d\bar{a}_x}{dB} + \lambda\frac{d\bar{a}_x}{d\lambda} = x\frac{d\bar{a}_x}{dx} - \bar{a}_x;$$

also

$$f(A, B, \lambda, x + h) = f(A, Be^{\lambda h}, \lambda, x),$$

so that

$$\lambda B\frac{d\bar{a}_x}{dB} = \frac{d\bar{a}_x}{dx};$$

we have therefore

$$\lambda\frac{d\bar{a}_x}{d\lambda} = \left(x - \frac{1}{\lambda}\right)\frac{d\bar{a}_x}{dx} + (A + \delta)(I\bar{a})_x - \bar{a}_x.$$

It is to be noted that these relations are obtained for continuous annuity values and are not strictly correct for annual annuities.

92. In some cases where a mortality table cannot be represented by Makeham's law in its simplest form without change of constants, we may put $l_x = l_x' \pm l_x''$ where l_x' and l_x'' each follow that law. In such cases it is of advantage to use if possible the same value of c in each of the subsidiary tables. The constants l_x' are determined from the experience at the older ages and the constants of l_x'' are then determined so as to fit the values of $\pm(l_x - l_x')$. It has also been suggested to modify the formula by adding an additional term and putting $\mu_x = A + Hx + Bc^x$, and Mr. Hardy found, in graduating the O^M table, that where the $O^{M(5)}$ table had been graduated by Makeham's law, he could use a relation of the form

$$\text{Colog}_{10}\,(p_x)^{O^M} = \text{colog}_{10}\,(p_x)^{O^{M(5)}} - \sum_x^\infty (Pe^{-a(x-e)^2} + Qe^{-b(x-m)^2}).$$

Or, in other words, the differences between the values of $\Delta\,\text{Colog}_{10}\,p_x$ according to the two tables was represented by a double frequency curve.

93. Another formula intended to cover the whole range of the mortality table from infancy to extinction was suggested by Wittstein. His assumption is

$$q_x = a^{-(M-x)^n} + \frac{1}{m}a^{-(mx)^n}.$$

Here M is one year less than the limiting age. To determine

m we have

$$\frac{dq_x}{dx} = n(M - x)^{n-1} \log a \, a^{-(M-x)^n} - n(mx)^{n-1} \log a \, a^{-(mx)^n}.$$

This evidently vanishes when $mx = M - x$ or $x = M/(m + 1)$. The age for which q_x is to be a minimum is then decided upon and m is determined by the relation $m + 1 = M/x$. Also we have $q_0 = (1/m) + a^{-M^n}$. The value of m should be so determined as to reconcile as well as possible these two considerations. The term $(1/m)a^{-(mx)^n}$ becomes negligible for adult ages and in order to determine the values of a and n we may put

$$q_x = a^{-(M-x)^n}$$

or

$$\log \log \frac{1}{q_x} = n \log (M - x) + \log \log a.$$

The respective terms are then properly weighted and the values of n and a determined by the method of moments or by the method of least squares. It is evident that if we denote $a^{-(M-x)^n}$ by q_x' and $(1/m)a^{-(mx)^n}$ by q_x'' we have $q_x = q_x' + q_x''$. Then for infantile ages where the second term preponderates we have

$$(q_x - q_x') = \frac{1}{m} a^{-(mx)^n}.$$

This may if desired be used to calculate by the method of moments or of least squares a corrected value of m. It was thought by Mr. Wittstein that the quantities a and n might prove to be absolute constants with a value of a in the neighborhood of 1.421 and of n in the neighborhood of .633. (42)

Comparison of Different Methods.

94. We have now four different graduations of the same mortality experience. We may therefore make a comparison of these graduations from the standpoint of smoothness and of agreement with the original data. From the standpoint of smoothness we take out the third differences of the rates of mortality and add together their numerical values regardless of sign. As, however, the summation graduation formula was modified at each extremity

over substantially a section of fourteen ages, we break the 42 third differences into three groups of 14 each and record the total of each group. The third differences over five-year intervals were also taken out and the thirty such differences for each graduation added together regardless of sign. This latter total is an indication of the extent to which major irregularities are left in the graduated table. The results are as follows:

	Sum of Third Differences.				
	Unit Interval.				5-Year Interval, Total.
	1st Group.	2d Group.	3d Group.	Total.	
Graphic............	.0030	.0048	.0190	.0268	.4921
Interpolation.......	.0019	.0163	.0280	.0462	.9814
Summation.........	.0032	.0137	.0090	.0259	.8080
Makeham..........	.0011	.0097	.0090	.0198	.1436

95. From the above comparison it appears that, as applied in these sample graduations, the graphic and summation methods are about equal in their effect on minor irregularities being each somewhat less powerful in this respect than Makeham's formula and more powerful than the interpolation method. When we examine major irregularities the Makeham graduation is, as was to be expected, again the most regular, the graphic method being next with third differences averaging 3½ times as great, the summation method nearly 6 times and the interpolation method, again last, 7 times as great. The order as regards smoothness is therefore, first Makeham's law, second, graphic method, third, summation method and, fourth, interpolation method. It is, in fact, evident that when Makeham's law is used the differences represent those arising from the law itself combined with those arising from irregularities due to dropped fractions. Under the graphic method the smoothness is, of course, limited only by the judgment of the graduator as to what is permissible in the way of departure from the original facts.

96. From the standpoint of agreement with data the following table shows the actual deaths and the deviations according to each graduated table by groups of ages. These groups are quinquennial, except that at each extremity the groups are combined so that in each group the actual number of deaths is in excess of ten.

Age Group.	Actual Deaths.	Deviations.			
		Graphic.	Interpolation.	Summation.	Makeham.
55–67..	16	+ 5.3	+ 5.7	+ 3.7	+10.3
68–72..	44	−10.7	− 9.2	− 6.6	− 4.9
73–77..	65	+ 1.7	+ 2.8	− .4	− .9
78–82..	92	+11.4	+11.2	+10.4	− 1.1
83–87..	112	−13.1	−19.9	− 9.3	−19.7
88–92..	49	+ 4.7	+ 3.3	+ 3.1	+ 9.2
93–99..	20	+ 1.2	+ 5.5	.0	+ 7.2
	398	±48.1	±57.6	±33.5	±53.3

97. We have, therefore, five items for comparison, (1) the sum of the deviations with regard to sign or the deviation in the total deaths; (2) the sum of the accumulated deviations, with regard to sign; (3) the sum of the individual deviations without regard to sign; (4) the sum of the accumulated deviations without regard to sign; (5) the sum of the group deviations without regard to sign. These are shown in the following table with a sixth column added, giving the expected or average value of the fifth item.

	(1)	(2)	(3)	(4)	(5)	(6)
Graphic.........	+.5	− 9.9	91.9	132.9	48.1	40.5
Interpolation....	−.6	−15.1	98.6	168.9	57.6	40.6
Summation......	+.9	+17.8	84.3	106.4	33.5	40.5
Makeham.......	+.1	+ 3.4	101.9	253.2	53.3	41.1

98. The figures in the first column are not significant, being such as might arise from dropped fractions. This is scarcely true of the second column, except for the last item, but even here the largest item represents a variation of less than one twentieth of a year in the average age at death. From the third and fourth columns it is seen that for individual ages the summation graduation agrees most closely with the original facts and is followed in order by the graphic, the interpolation and the Makeham graduations. The group deviations by the interpolation graduation are, however, greater than by the Makeham graduation. Comparing columns (5) and (6) the group deviations appear to be less in the summation graduation than the expected and greater in the others. If we examine the column of accumulated deviations in the comparative tables given under each graduation

we find 8 changes of sign for the graphic, 9 changes for the interpolation, 10 changes for the summation and 8 changes for the Makeham graduation. The variations in this respect are therefore not material.

99. Taking all these considerations into account it would appear that the Makeham graduation must in this case be considered unsatisfactory on account of the large accumulated deviations shown in column (4) and the marked excess of the group deviations over the expected. So also must the interpolation graduation on account of the excess of group deviations combined with the deficiency in smoothness. In the choice between the other two it is to be noted that, although the sum of the third differences is practically the same in the graphic graduation as in the summation, the greater part of the former total arises from the least important group. It would, in fact, have been greatly reduced had values of q_x, derived from the formula, been used at the older ages in the standard table instead of those derived from the l_x column. This fact combined with the smaller third differences over five-year intervals gives the preference on the score of smoothness to the graphic graduation. The fact that the group deviations in the summation graduation are materially less than the expected also indicates that it follows too closely the major irregularities of the original data. The preference on the whole, falls therefore to the graphic graduation, although the excess of the group deviations over the expected indicates that it could be improved by bringing it more closely into harmony with the original data in those sections where the deviation is the greatest.

100. In considering the method of graduation to be adopted in any particular case, whether the series to be graduated is a frequency distribution or a series of ratios, as in the case of a mortality experience, it is evident that a graduation by mathematical formula possesses an advantage over all others on the score of smoothness. And where at least two arbitrary constants are available the sum of the deviations and the accumulated deviations can be made to vanish. In view of these advantages a mathematical formula will be the best provided two further conditions are satisfied. The first is that the total irrespective of sign of the deviations, in suitable groups, should not materially exceed the expected. The amount of excess which is permissible will

depend on the number of groups used in the comparison. The second condition is that any special feature known to be characteristic of the series should be reproduced by the formula. In deciding this point an examination of allied series should be made to see if the same special feature occurs. If it does not and if there is no assignable cause, other than accidental fluctuation, for the appearance of the special feature in the series in question, it should be considered as entirely accidental and the mathematical formula used even though it does not reproduce it. For the graduation of mortality tables, Makeham's law possesses additional advantages owing to its special adaptation to the calculation of joint and contingent benefits in connection with insurance or annuity transactions. For any table, therefore, which it is expected to use for such purposes that law should be used if possible, a very liberal interpretation being given to the two conditions above mentioned.

101. Where a mathematical law cannot be applied it will usually be found that where the data are very scanty the graphic method will produce the best results as irregularities will occur of wide range, such as neither the interpolation nor the summation method is competent to remove. The interpolation method may be used, however, in combination with the graphic, the latter being used instead of Mr. King's formula to determine the points upon which to interpolate. The points will, of course, be subject to subsequent modification if necessary, just as the curve is subject to subsequent amendment in the regular graphic method. This amounts to the substitution of an analytical interpolation for the graphic between the selected points.

102. Where, however, the data are more extensive so as to give a satisfactory degree of regularity under the operation of the interpolation or of the summation method, those methods will be the more satisfactory as the values derived do not depend on the judgment of the operator except as exercised in the selection of the particular graduation formula to be used, and they can be obtained to a greater degree of accuracy than is possible in reading them from a diagram. As between the two methods, interpolation will probably be found the more useful in connection with census data and other cases where the original facts are given in groups. In other cases a combination of the two methods may be sometimes

used, the points for the interpolation being determined by means of a summation formula of great weight without regard to its smoothing coefficient and the interpolation being depended upon to introduce the necessary smoothness.

Note.

In the course of this study we have had occasion to refer to certain theorems regarding the results of repeated trials, the proof of which may not be available for reference by the student. They are accordingly collected in this note. Where the probability of an event happening at each trial is p and that of its failing is $q = 1 - p$ the probability that in n trials it will happen x times and fail $n - x$ times in any assigned order is evidently $p^x q^{n-x}$. As, therefore, there are in all $\underline{|n}/(\underline{|x}\,\underline{|n - x})$ different ways in which the event may happen x times and fail $n - x$ times the total probability of exactly x successes out of n trials is

$$\frac{\underline{|n}}{\underline{|x}\,\underline{|n - x}} p^x q^{n-x}.$$

To determine then the expected number of successes we multiply each number by its probability and sum, the result being

$$\begin{aligned}\sum_1^n x \frac{\underline{|n}}{\underline{|x}\,\underline{|n - x}} p^x q^{n-x} &= np \sum_1^n \frac{\underline{|n - 1}}{\underline{|x - 1}\,\underline{|n - x}} p^{x-1} q^{n-x} \\ &= np(p + q)^{n-1} = np.\end{aligned}$$

Similarly the mean value of $x(x - 1)$ is equal to

$$\begin{aligned}&\sum_2^n x(x - 1) \frac{\underline{|n}}{\underline{|x}\,\underline{|n - x}} p^x q^{n-x} \\ &\qquad = n(n - 1)p^2 \sum_1^n \frac{\underline{|n - 2}}{\underline{|x - 2}\,\underline{|n - x}} p^{x-2} q^{n-x} = n(n - 1)p^2\end{aligned}$$

and generally the mean value of $x(x - 1)(x - 2)\cdots(x - r + 1)$ is $n(n - 1)(n - 2)\cdots(n - r + 1)p^r$.

Thus in the notation of moments we have, taking zero as origin,

$$\begin{aligned} m_1 &= np, \\ m_2 - m_1 &= n(n - 1)p_2, \\ m_3 - 3m_2 + 2m_1 &= n(n - 1)(n - 2)p^3, \\ m_4 - 6m_3 + 11m_2 - 6m_1 &= n(n - 1)(n - 2)(n - 3)p^4. \end{aligned}$$

Whence

$$
\begin{aligned}
m_2 &= n^2p^2 + npq, \\
m_3 &= n^3p^3 + 3n^2p^2q + npq(q - p), \\
m_4 &= n^4p^4 + 6n^3p^3q - n^2(11p^2 - 18p^2q - 11p^4) \\
&\qquad\qquad\qquad + n\{6p - 11pq + 6pq(q - p) - 6p^4\} \\
&= n^4p^4 + 6n^3p^3q - n^2p^2q(4p - 7q) + npq(1 - 6pq).
\end{aligned}
$$

Or transferring to the mean value as origin

$$
\begin{aligned}
\mu_2 &= npq, \\
\mu_3 &= npq(q - p), \\
\mu_4 &= npq(1 - 6pq) + 3n^2p^2q^2.
\end{aligned}
$$

To determine the mean value of the departure from the expected number, without regard to sign, we must make two separate summations. If we designate by l the largest integer contained in np we have for this mean value

$$
\begin{aligned}
&\sum_0^l (np - x)\frac{\lfloor n}{\lfloor x \lfloor n - x} p^x q^{n-x} + \sum_{l+1}^n (x - np)\frac{\lfloor n}{\lfloor x \lfloor n - x} p^x q^{n-x} \\
&= \sum_0^l \{(n - x)p - xq\}\frac{\lfloor n}{\lfloor x \lfloor n - x} p^x q^{n-x} \\
&\qquad\qquad + \sum_{l+1}^n \{xq - (n - x)p\}\frac{\lfloor n}{\lfloor x \lfloor n - x} p^x q^{n-x} \\
&= \sum_0^l \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} - \sum_1^l \frac{\lfloor n}{\lfloor x - 1 \lfloor n - x} p^x q^{n-x+1} \\
&\qquad + \sum_{l+1}^n \frac{\lfloor n}{\lfloor x - 1 \lfloor n - x} p^x q^{n-x+1} - \sum_{l+1}^{n-1} \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} \\
&= \sum_0^l \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} - \sum_0^{l-1} \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} \\
&\qquad + \sum_l^{n-1} \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} - \sum_{l+1}^{n-1} \frac{\lfloor n}{\lfloor x \lfloor n - x - 1} p^{x+1} q^{n-x} \\
&= 2\frac{\lfloor n}{\lfloor l \lfloor n - l - 1} p^{l+1} q^{n-l}.
\end{aligned}
$$

In order to reduce this expression to a simple form it is necessary to substitute an approximate expression for the factorials involved.

We have

$$\log_e \frac{n}{n-1} = 2\left\{\frac{1}{2n-1} + \frac{1}{3}\frac{1}{(2n-1)^3} + \frac{1}{5}\frac{1}{(2n-1)^5} + \cdots\right\},$$

whence

$$\left(n - \frac{1}{2}\right)\log_e \frac{n}{n-1} - 1 = \frac{1}{3}\frac{1}{(2n-1)^2} + \frac{1}{5}\frac{1}{(2n-1)^4} + \cdots.$$

This is positive for all values of n and less than

$$\frac{1}{3}\left\{\frac{1}{(2n-1)^2} + \frac{1}{(2n-1)^4} + \cdots\right\} = \frac{1}{3(2n-1)^2}\left\{1 - \frac{1}{(2n-1)^2}\right\}^{-1}$$
$$= \frac{1}{12n(n-1)} = \frac{1}{12(n-1)} - \frac{1}{12n}.$$

But

$$(n - \tfrac{1}{2})\log_e \frac{n}{n-1} - 1$$
$$= (n + \tfrac{1}{2})\log_e n - (n - \tfrac{1}{2})\log_e (n-1) - \log_e n - 1$$
$$= \{(n + \tfrac{1}{2})\log_e n - \log_e \lfloor n - n\}$$
$$- \{(n - \tfrac{1}{2})\log_e (n-1) - \log_e \lfloor (n-1) - (n-1)\}.$$

Designating now $\{(n + \frac{1}{2})\log_e n - \log_e \lfloor n - n\}$ by $f(n)$ we see from this equation that $\Delta f(n-1)$ is always positive and less than $\left\{\frac{1}{12(n-1} - \frac{1}{12n}\right\}$. For any value of n greater than unity we have therefore since $f(1) = -1$

$$-1 < f(n) < -1 + \frac{1}{12} - \frac{1}{12n} < -\frac{11}{12}.$$

When n is indefinitely increased $f(n)$ must tend to a definite limit which we may designate by $-c$. For any finite value of n the value of this function must differ from its ultimate value $-c$ by less than

$$\sum_n^{\infty}\left\{\frac{1}{12n} - \frac{1}{12(n+1)}\right\} = \frac{1}{12n}.$$

We have therefore

$$(n + \tfrac{1}{2})\log_e n - \log_e \lfloor n - n = -c - \frac{\theta}{12n},$$

where $0 < \theta < 1$ or

$$\log_e \lfloor n = (n + \tfrac{1}{2})\log_e n - n + c + \frac{\theta}{12n},$$

where the last term may be neglected if n is large. The exact value of c is determined from the fact that $\pi/2$ is the limit, when n is indefinitely increased, of $\dfrac{2^{2n}(\lfloor n)^2(2n+1)}{1^2\cdot 3^2\cdots(2n+1)^2}$ or of $\dfrac{2^{4n}(\lfloor n)^4(2n+1)}{(\lfloor 2n+1)^2}$.
Hence $\log_e \pi$ is the limit of
$(4n+1)\log_e 2 + 4\log_e \lfloor n - 2\log_e \lfloor 2n - \log_e(2n+1)$

$$= (4n+1)\log 2 + 2(2n+1)\log_e n - 4n + 4c$$
$$- (4n+1)\log_e 2n + 4n - 2c - \log_e(2n+1)$$
$$= \log_e n + 2c - \log_e(2n+1) = 2c - \log_e 2, \text{ in limit.}$$

Hence

$$2c = \log_e 2\pi,$$
$$c = \tfrac{1}{2}\log_e 2\pi.$$

Hence we have

$$\lfloor n = \sqrt{2\pi}\,\frac{n^{n+(1/2)}}{e^n},$$

when n is large.

Returning then to the expression

$$2\,\frac{\lfloor n}{\lfloor l\,\lfloor n-l-1}\,p^{l+1}q^{n-l} \quad \text{or} \quad \frac{2\,\lfloor n\,(n-l)}{\lfloor l\,\lfloor n-l}\,p^{l+1}q^{n-l}$$

for the mean departure and putting $l = np - k$, where k is fractional, and consequently $n - l = nq + k$ the expression takes the form, when the approximate values of the factorials are used,

$$\frac{2}{\sqrt{2\pi}}\,n^{n+\frac{1}{2}}(nq+k)(np-k)^{-np-\frac{1}{2}+k}(nq+k)^{-nq-\frac{1}{2}-k}p^{np+1-k}q^{nq+k}$$

$$= \sqrt{\frac{2npq}{\pi}}\left(1-\frac{k}{np}\right)^{-np-\frac{1}{2}+k}\left(1+\frac{k}{nq}\right)^{-nq+\frac{1}{2}-k}$$

$$= \sqrt{\frac{2npq}{\pi}}\,e^k\cdot e^{-k}$$

(approximately) since k is small compared with np or nq.

Hence the expression reduces to

$$\sqrt{\frac{2npq}{\pi}} = .79788\sqrt{npq} = \tfrac{4}{5}\sqrt{npq}.$$

Again, putting $np + x$ for x in the expression for the probability of exactly x successes in n trials and so transferring the origin

to the expected number it becomes

$$\frac{\underline{|n}}{\underline{|np+x}\;\underline{|nq-x}}\, p^{np+x}q^{nq-x}.$$

Substituting then the approximate values of the factorials this becomes

$$\frac{1}{\sqrt{2\pi}}\, n^{n+\frac{1}{2}}(np+x)^{-np-x-\frac{1}{2}}(nq-x)^{-nq+x-\frac{1}{2}}p^{np+x}q^{np-x}$$

$$= \frac{1}{\sqrt{2\pi npq}}\left(1+\frac{x}{np}\right)^{-np-x-\frac{1}{2}}\left(1-\frac{x}{nq}\right)^{-nq+x-\frac{1}{2}}.$$

If now we suppose np and nq to be very large so that x/np may be neglected, but that x^2/np may not be neglected, the logarithm of this expression becomes

$$-\tfrac{1}{2}\log 2\pi npq - (np+x+\tfrac{1}{2})\log\left(1+\frac{x}{np}\right)$$

$$- (nq-x+\tfrac{1}{2})\log\left(1-\frac{x}{nq}\right)$$

$$= -\tfrac{1}{2}\log 2\pi npq - (np+x+\tfrac{1}{2})\left(\frac{x}{np}-\frac{x^2}{2n^2p^2}+\cdots\right)$$

$$+ (nq-x+\tfrac{1}{2})\left(\frac{x}{nq}+\frac{x^2}{2n^2q^2}+\cdots\right)$$

$$= -\tfrac{1}{2}\log 2\pi npq - \frac{x^2}{2np} - \frac{x^2}{2nq} = -\tfrac{1}{2}\log 2\pi pq - \frac{x^2}{2npq}$$

and the expression itself becomes

$$\frac{1}{\sqrt{2\pi npq}}\, e^{-x^2/(2npq)}.$$

Where np and nq are both large therefore the curve $y = \dfrac{1}{c\sqrt{\pi}}\, e^{-x^2/c^2}$ approximately represents the probabilities of the various departures when $c^2 = 2npq$ and from tables which have been formed of the integral of this expression it is found that there is approximately an even chance of the departure exceeding $2/3\,\sqrt{npq}$.

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